ECE 20875
Python for Data Science
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Probability and Random Variables
what is a probability?

• Measure of likelihood that an event occurs
• A number between 0 and 1
• The higher the number, the more likely the event occurs
• A probability of 0 means the event never occurs, and a probability of 1 means the event always occurs
• Example: What is the probability of the event “heads” when flipping a coin? 
  \[ P(H) = \]
elements of a probability model

- Conduct an **experiment**, which results in an **outcome**
- Each outcome has a probability between 0 and 1
- Set of all possible outcomes is the **sample space** $\Omega$
- Sum of probability of all outcomes is 1
- An **event** is a set of possible outcomes
  - Probability of event is the sum of the probabilities of individual outcomes

$$\Omega = \{1,\ldots,5\}$$

$$P(3) = \frac{3}{8}$$

$$P(\{1,3,5\}) = \frac{5}{8}$$
Probability

Here is a picture about probability:

I You do an experiment.
I You collect outcomes, a point in the sample space.
I A sub-collection of outcomes is called an event.
I Then you assign different events to probability.
what does probability mean?

- Lots of different interpretations

- All outcomes are equally probable (e.g., roll a die, each number has the same chance). Probability of an event is number of outcomes in event divided by total number of outcomes.

- **Frequentist**: Repeat an experiment over and over again, probability of an event is fraction of the time the event happens during the experiment.

- **Bayesian**: Probability is a reflection of your belief about the likelihood of something happening (e.g., based on prior knowledge).
random variables

- A **random variable** $X$ is a function that assigns an outcome to a number
  - A way of letting us treat outcomes, which may not be numbers, in a mathematical way
  - E.g., in flipping a coin, $X$ could map Heads to 0 and Tails to 1
- A random variable has a probability distribution which tells us the probability of its values
  - E.g., in flipping a coin, $P[X = 0] = 0.5$, $P[X = 1] = 0.5$
- Informal intuition: The random variable is the horizontal value on the histogram, with the height being the probability
- Random variables can be **continuous** or **discrete**
probability density function

• One loose definition: A histogram when …
• (i) the number of samples goes to infinity
• (ii) the bin width approaches zero
• When this happens, the estimate \( \hat{p}_k \) approaches \( p_k \) of the population
• More formal definition: \( f_X(x) \) is the **probability density function** (PDF) for \( X \) if

\[
P[a \leq X \leq b] = \int_a^b f_X(x) \, dx
\]

• \( X \) is a continuous random variable
The **cumulative distribution function** (CDF) of a random variable $X$ is

$$F_X(x) = P[X \leq x]$$
probability mass/density function

• If $X$ is a discrete random variable, it has a **probability mass function (PMF)**. The PMF is defined directly from the probabilities of events (essentially a histogram with bars interpreted as frequencies):
  \[ f_X(x) = P[X = x] \]

• If $X$ is a continuous random variable, it has a PDF, which is a little trickier to define since the probability of any single number is actually 0. As a result, we can also define the PDF in terms of the CDF:
  \[ f_X(x) = \frac{dF_X(x)}{dx} \]
CDF from PDF/data

- The **continuous CDF** \( F_X(x) \) in terms of the PDF \( f_X(x) \):
  \[
  F_X(x) = P[X \leq x] = P[-\infty \leq X \leq x] = \int_{-\infty}^{x} f_X(t)dt
  \]

- The **discrete CDF** \( F_X(x) \) in terms of the PMF \( f_X(x) = P[X = x] \):
  \[
  F_X(x) = P[X \leq x] = P[-\infty \leq X \leq x] = \sum_{x_i \leq x} f_X(x_i) = \sum_{x_i \leq x} P[X = x_i]
  \]
  where \( x_i \) are possible discrete values (e.g., 0, 1, 2, …)

- For a dataset of \( n \) points, we can define a **discrete empirical CDF**:
  \[
  F_X(x) = P[X \leq x] = P[-\infty \leq X \leq x] = \sum_{x_i \leq x} f_X(x_i) = \sum_{x_i \leq x} \frac{1}{n}
  \]
  where \( x_i \) are the samples (e.g., height in feet 5.8, 6.1, 5.1, …)

- Note that each of these functions are defined for all values of \( x \), even though the random variables may be continuous or discrete!
picking a distribution

- Common problem in data science
- You have (empirical) data, and you need to choose how to (analytically) model it
- What distribution is your data coming from?
- What distribution is most likely to predict future samples?
- Important choice because distribution often determines how your model works
qq plots

- Basic idea: Compare the empirical CDF of your data to the CDF of a proposed model
- Use quantiles to do this ("inverse" of CDF function)
  - Quantile $q$ is the value of $x$ such that $P[X \leq x] = q$
  - Sometimes expressed in terms of percentiles, e.g., scoring in the 95th percentile on a test
- For each datapoint in your sample, find:
  - The quantile with respect to the dataset, $q_D$
  - The quantile with respect to the model, $q_M$
- Add each point $(q_M, q_D)$ to a scatter plot
  - If the distributions are similar, the quartiles will appear to form the line $y = x$
qq plots

- See `scipy.stats.probplot`
- Axis values here are in terms of standard deviations from the mean
beroulli distribution

- Two states: $X = 0$ or $X = 1$
- Think flipping a coin, or a single “bit” of information
- But it doesn’t have to be a fair coin!
- PMF:

$$P[X = x] = \begin{cases} 1 - p & x = 0 \\ p & x = 1 \end{cases}$$

- Here, $p \in [0,1]$ is the probability of “success” (i.e., $X = 1$)
binomial distribution

• Bernoulli trials repeated \( n \) times

• Think flipping a coin \( n \) times and counting the number of heads, or transmitting \( n \) bits and counting the number of 1’s

• PMF:

\[
P[X = x] = \binom{n}{x} p^x (1 - p)^{n-x}
\]

• Here, \( \binom{n}{x} = \frac{n!}{x!(n-x)!} \) is the binomial coefficient
We are interested in modeling whether a machine produces outputs in spec or not.

We collect 200 samples and find 20 are out of spec.

Model the next output as a random variable.

What is its pmf?
Let $X = 0$ denote “out of spec” and $X = 1$ denote “in spec”.

$X$ is a Bernoulli random variable, and from the data, we can estimate $p = 180/200 = 0.9$ as the probability of success.

Hence,

$$f_X(x) = \begin{cases} 
0.1, & x = 0 \\
0.9, & x = 1 
\end{cases}$$

$$F_X(x) = \begin{cases} 
0, & x < 0 \\
0.1, & 0 \leq x < 1 \\
1, & x \geq 1 
\end{cases}$$
gaussian distribution

- Also called the **normal** distribution, or the bell curve
- Very common distribution in natural processes
- The sum of many independent processes is often normal (more on this later)
- PDF:
  \[ \mathcal{N}(x | \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \]
- Its parameters are the **mean** \( \mu \) and the **variance** \( \sigma^2 \)
The PDF of the normal distribution has several useful properties.

- **The 3-sigma rule**
  - ~68% of points within ±σ of μ
  - ~95% of points within ±2σ of μ
  - ~99.7% of points within ±3σ of μ

- Useful in constructing confidence intervals and hypothesis testing (more on this later)
exponential distribution

- Useful for modeling decay processes, inter-arrival times, and occurrences of events
- Probability of a radioactive item decaying
- Time between arrival of visitors to a website, or customers to a store

- PDF:
  \[ f_X(x) = \begin{cases} 
  \lambda e^{-\lambda x} & x \geq 0 \\
  0 & x < 0 
  \end{cases} \]

- \( \lambda > 0 \) is the rate parameter
We are told that the time between visits to a website, measured in minutes, is exponentially distributed with a rate parameter $\lambda = 2$. 

1) Find the CDF of this random variable.

2) What is the probability that there is more than 0.5 minutes between visits?
The random variable $X$ has the following PDF:

$$f_X(x) = \begin{cases} 0, & x < 0 \\ 2e^{-2x}, & x \geq 0 \end{cases}$$

We can find the CDF as:

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt = \int_{0}^{x} 2e^{-2t}dt = -e^{-2t} \bigg|_{0}^{x} = \begin{cases} 0, & x < 0 \\ 1 - e^{-2x}, & x \geq 0 \end{cases}$$

The probability of $X > 0.5$ is:

$$P[X > 0.5] = 1 - F_X(0.5) = 1 - (1 - e^{-2(0.5)}) = e^{-1} = 0.368$$
many more!

- Geometric: “How many times do I need to flip a coin to get heads?”
- Uniform: Every event in an interval is equally likely
- Student’s t: Behavior of normal distribution with fewer samples
- Poisson: Discrete version of the exponential distribution
- …