ECE 20875
Python for Data Science

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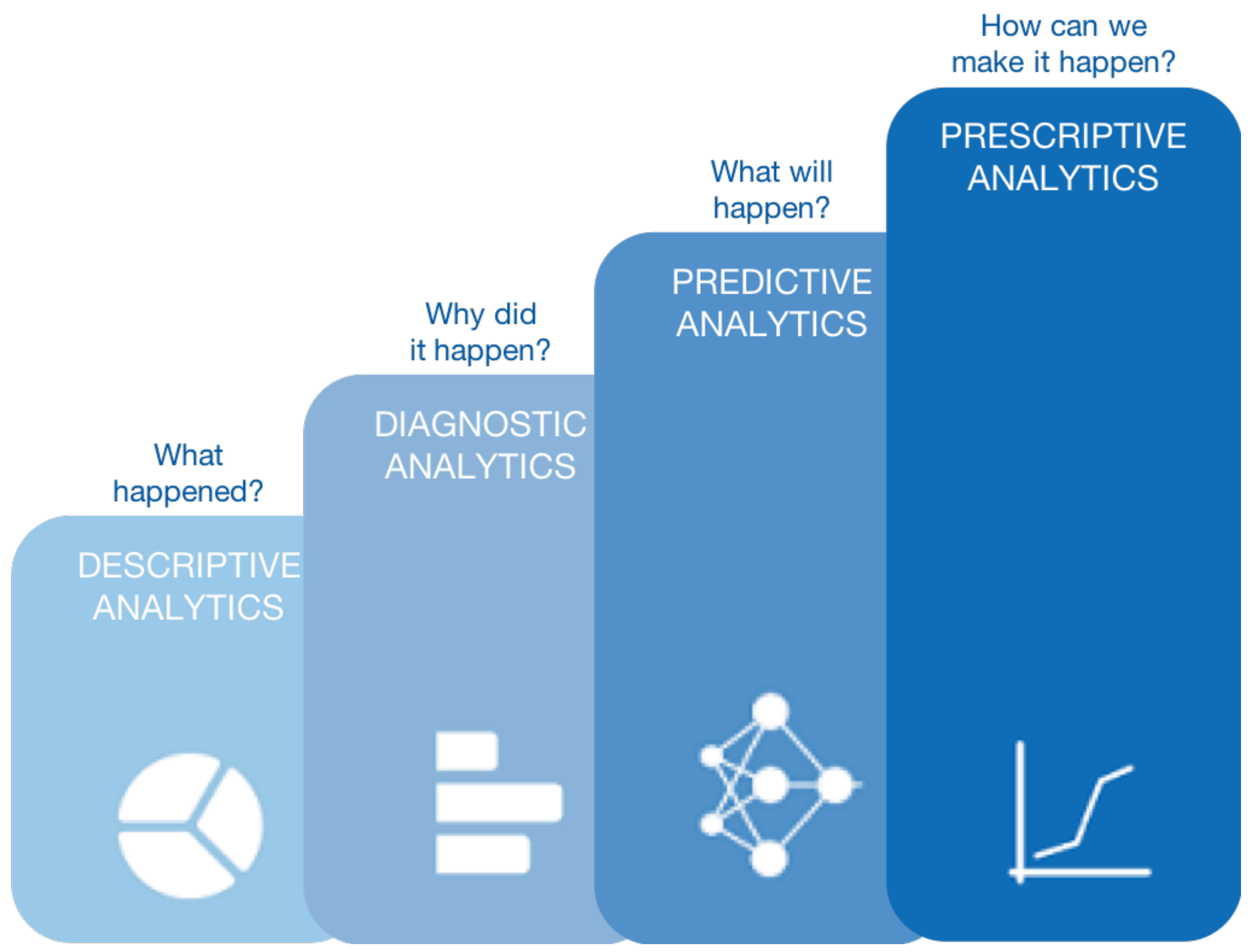
regression

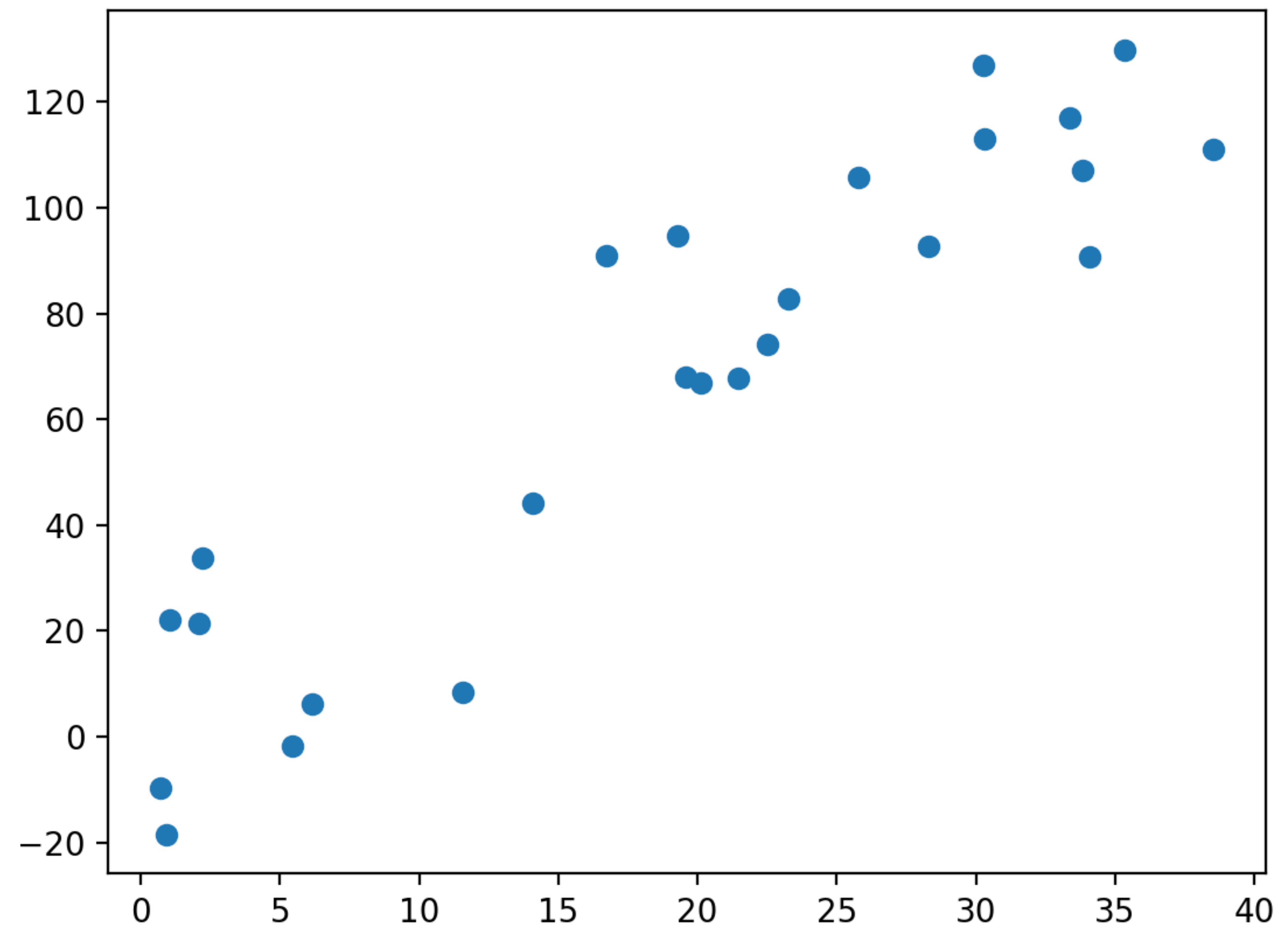
inference

- **Inference** is one of the basic problems that we want to solve in data science
  - Given a set of data that we know some facts about, what new conclusions can we draw, and with what certainty?
  - We will investigate several approaches to drawing conclusions from given sets of data
- Over the next few lectures: Making **predictions** about new data points given existing data using **linear regression**



linear regression

- Basic modeling problem: I want to identify a relationship between …
    - **explanatory variables** (i.e., the “inputs”, often referred to as the **features** of a data point), and
    - a **target variable** (i.e., some “output” quantity that we want to estimate)
- Can we learn what this relationship is?
- If we have a **model** for this relationship, we can use it to predict the target variable for new data points



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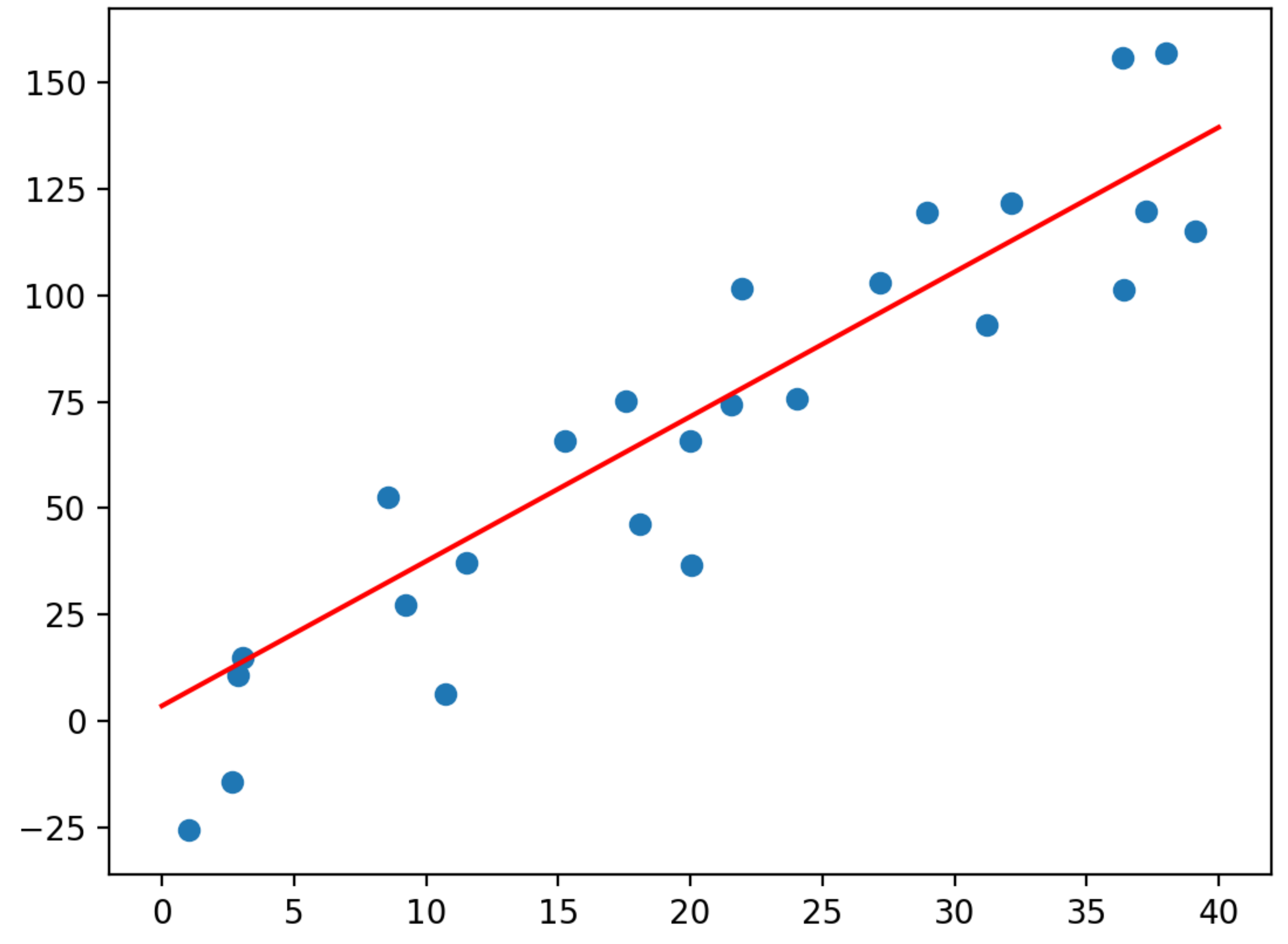
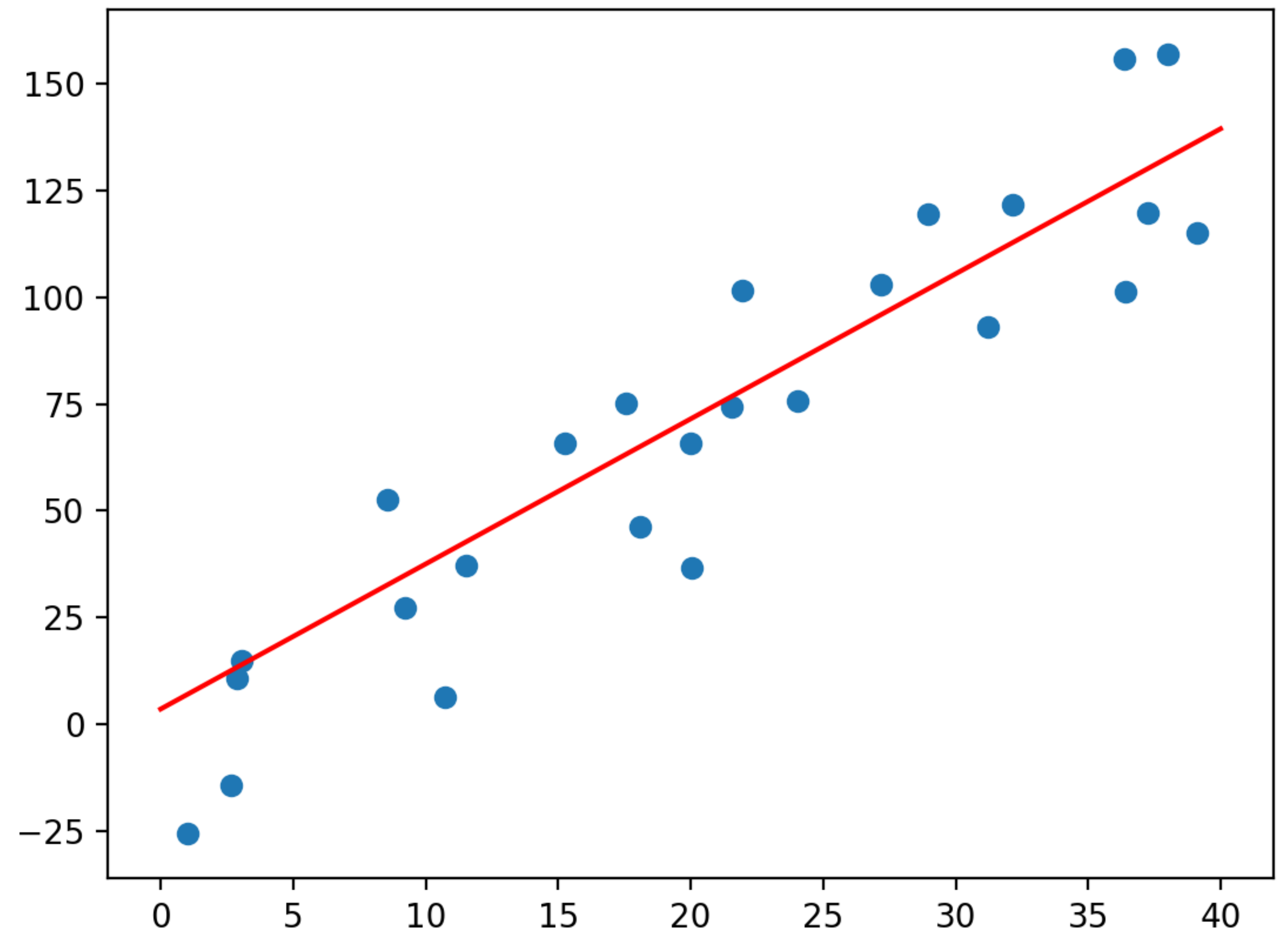


linear regression

- Can we learn the model from the data?
- Note that the model does not match the data exactly!
  - A model is (at best) a simplification of the real-world relationship
- What makes a good model?
  - Minimizes **observed error**: How far the model deviates from the observed data
  - Maximizes **generalizability**: How well the model is expected to hold up to unseen data



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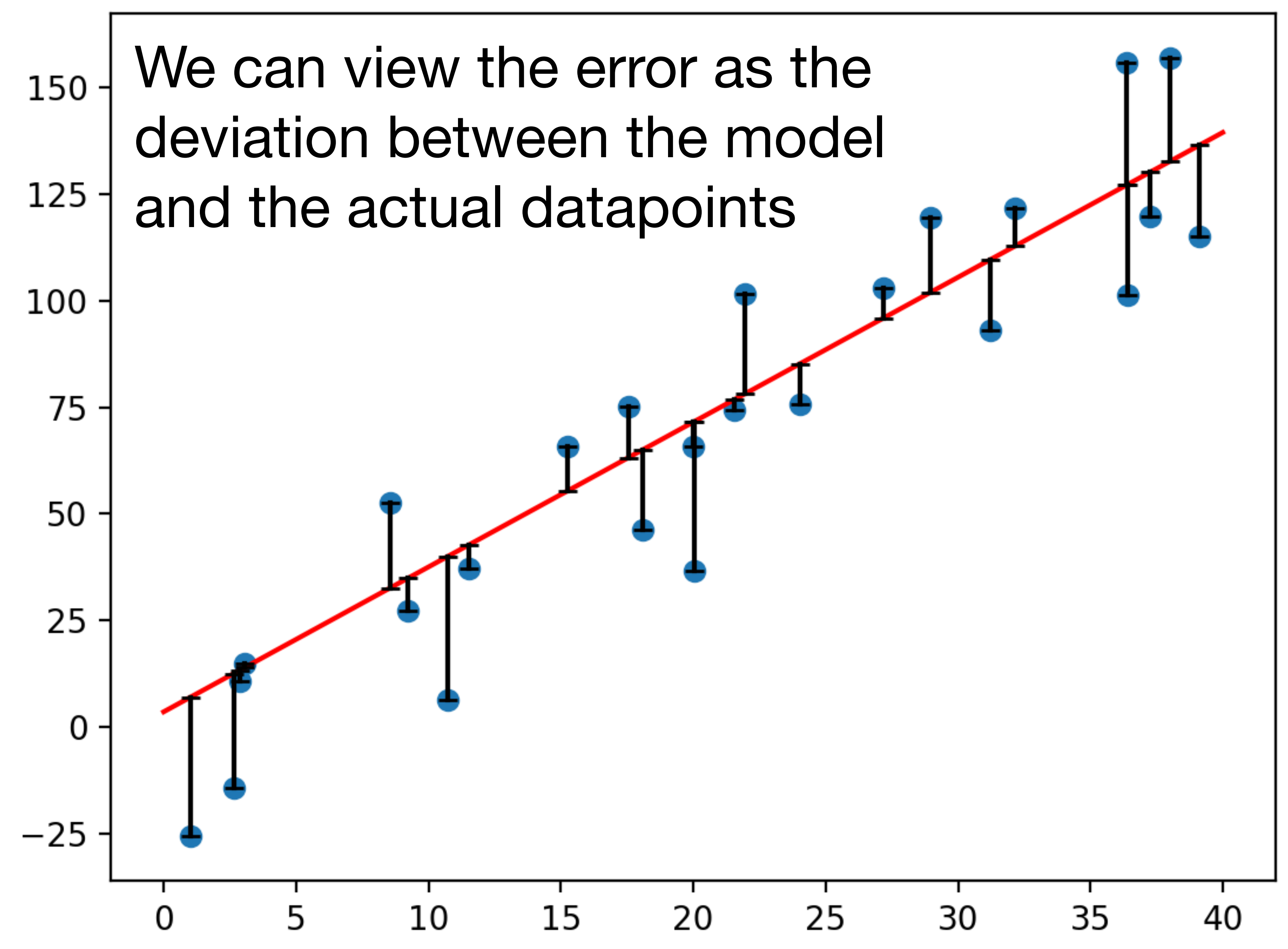
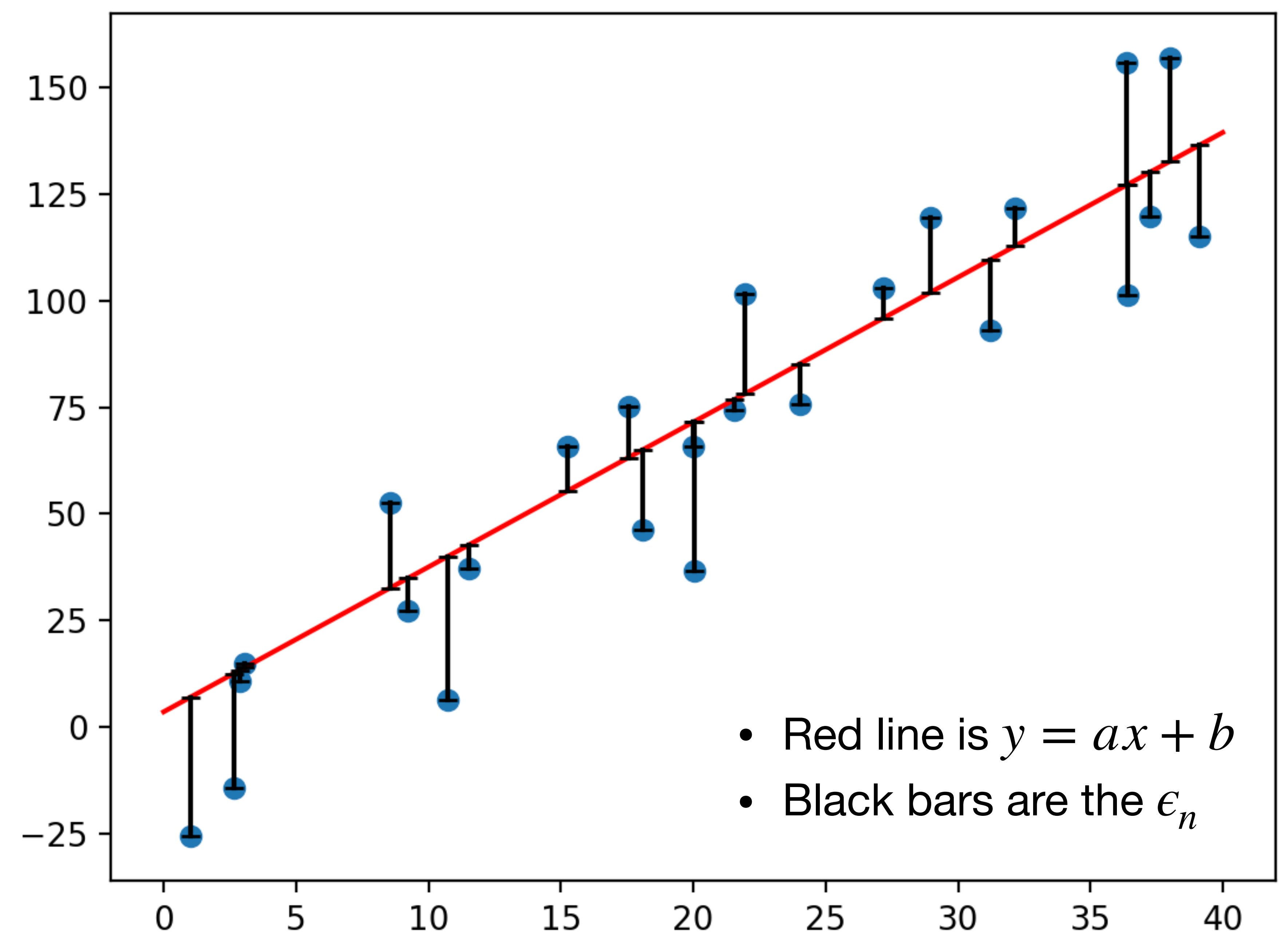


simple linear regression model

- The **simple linear regression** model has a single explanatory variable:
  $y_n = ax_n + b + \epsilon_n, \;\; n = 1,\ldots,N$
- y_n is the **measured value** of the target variable for the nth data point
- $ax_n + b$ is the **estimated value** of the target, based on the explanatory x_n
- Each y_n is associated with a model prediction component $ax_n + b$ plus some **error term** ϵ_n
- How do we minimize this error?



minimizing error

- The **mean squared error** (MSE) for simple linear regression is

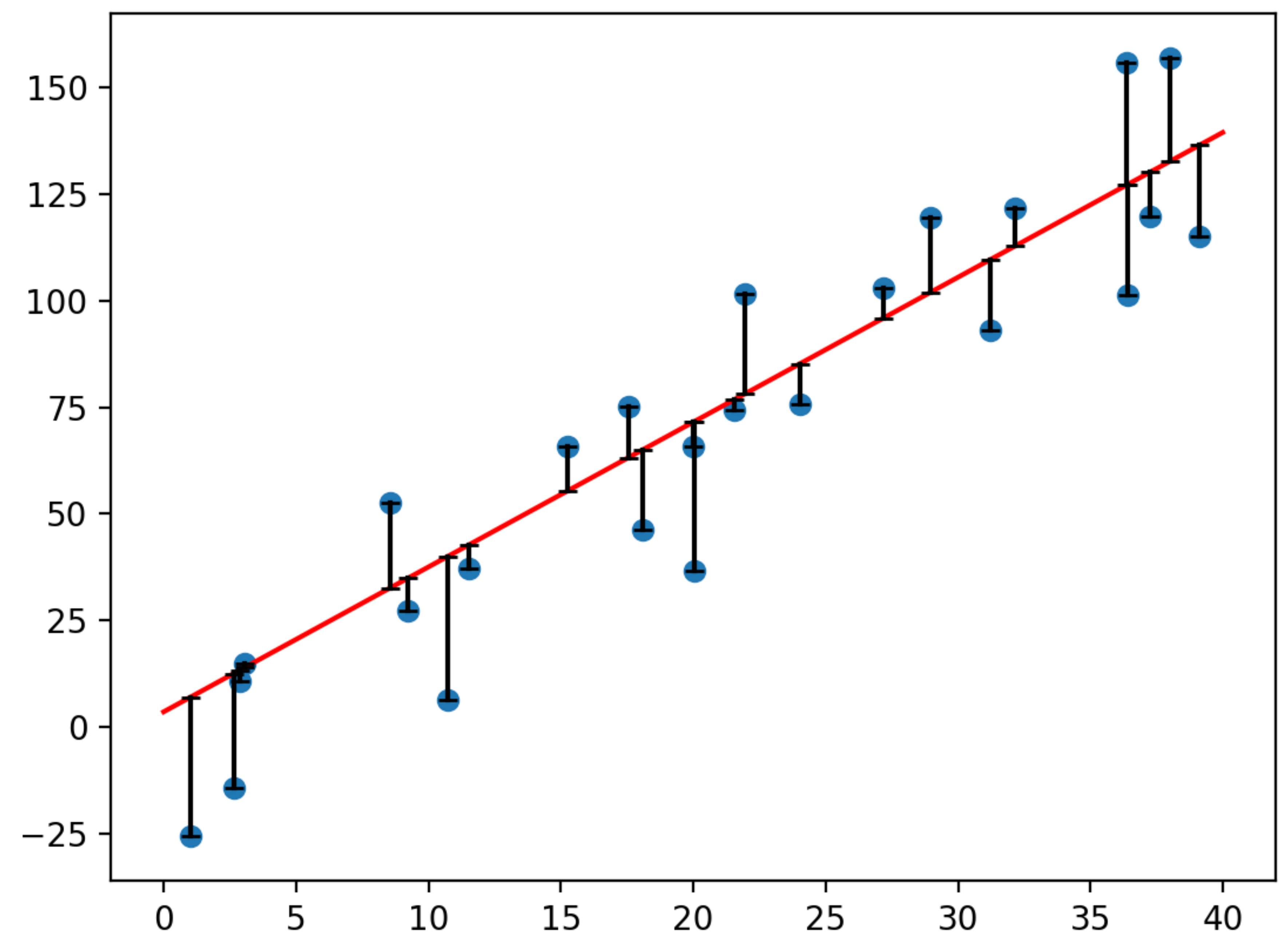
$$E(a, b) = \frac{1}{N} \sum_{n=1}^{N} \left(y_n - (ax_n + b) \right)^2$$

- Common error metric: We looked at already when we studied the choice of histogram bin widths

- We want to minimize E, denoted: $\min_{a,b} E(a, b)$

  - With two **model parameters** a and b, this is reasonably easy to carry out by hand

  - The square makes it easy to take the derivative



minimizing error: derivation

- Set the derivatives with respect to a and b to zero:

$$\frac{dE}{da} = \frac{1}{N}\sum_{n=1}^{N} -2x_n\left(y_n - (ax_n + b)\right) = 0$$

$$\frac{dE}{db} = \frac{1}{N}\sum_{n=1}^{N} -2\left(y_n - (ax_n + b)\right) = 0$$

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- Focusing first on the second equation, we have:

$$\frac{-\sum_{n=1}^{N} y_n}{N} + a\frac{\sum_{n=1}^{N} x_n}{N} + b\frac{\sum_{n=1}^{N} 1}{N} = 0, \text{ or}$$

$$b = \frac{\sum_{n=1}^{N} y_n}{N} - a\frac{\sum_{n=1}^{N} x_n}{N} = \bar{y} - a\bar{x}$$

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- Substituting our expression for b, we have:

$$a\frac{\sum_{n=1}^{N} x_n^2}{N} = \frac{\sum_{n=1}^{N} x_n y_n}{N} - (\bar{y} - a\bar{x})\bar{x}, \text{ or}$$

$$a\left(\frac{\sum_{n=1}^{N} x_n^2}{N} - \bar{x}^2\right) = \frac{\sum_{n=1}^{N} x_n y_n}{N} - \bar{y}\bar{x}$$

minimizing error: formulas

- Isolating a on the left hand side and simplifying, we get:

$$a = \frac{\sum_{n=1}^{N} x_n y_n - N\bar{y}\bar{x}}{\sum_{n=1}^{N} x_n^2 - N\bar{x}^2}$$

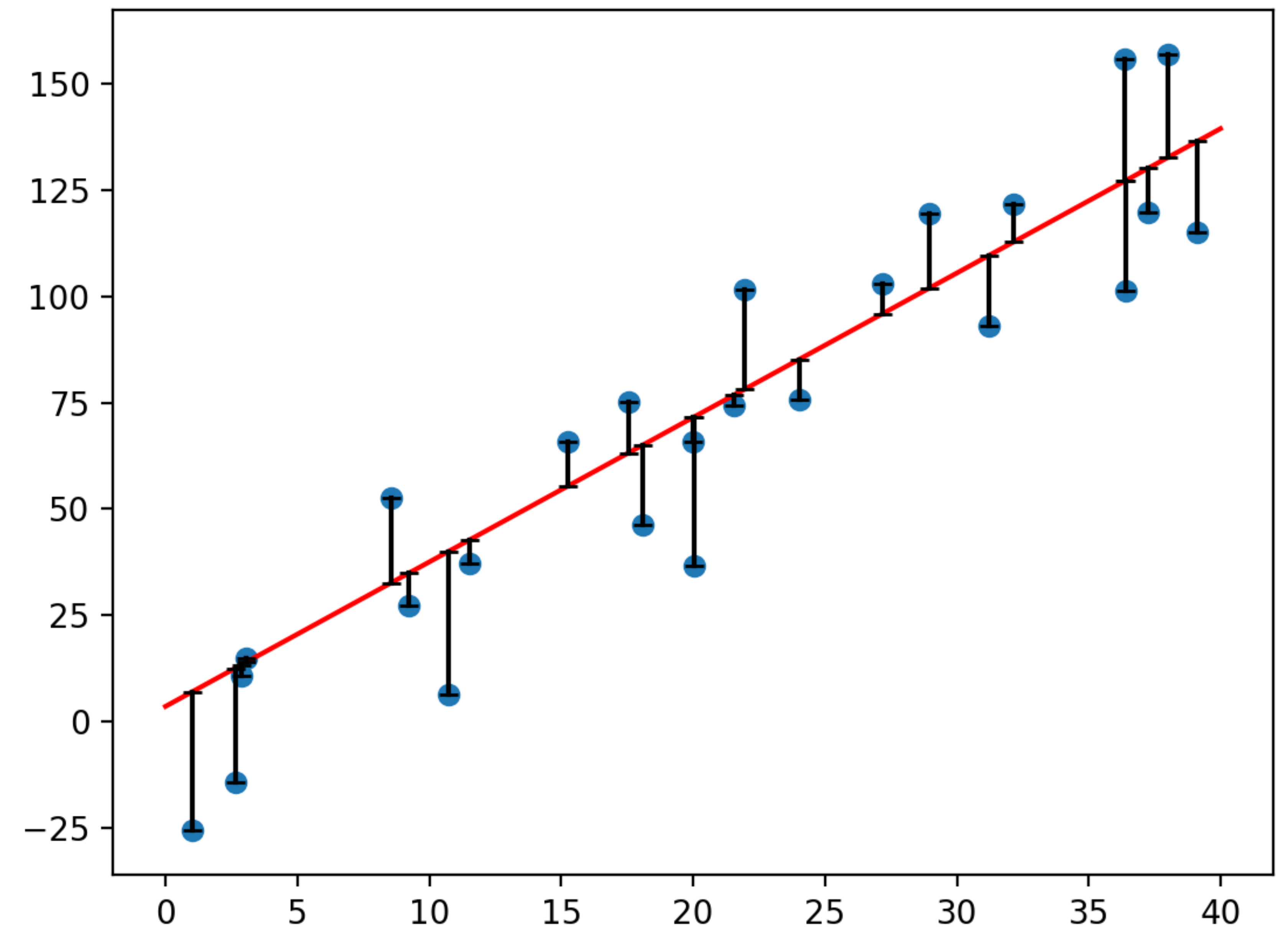
  - Here, $\bar{x}$ and $\bar{y}$ are the averages of the x_n and y_n, respectively

- We can then use a to solve for b according to:

$$b = \bar{y} - a\bar{x}$$

- And then our linear regression predictor for a new datapoint i is

$$y_i = ax_i + b$$



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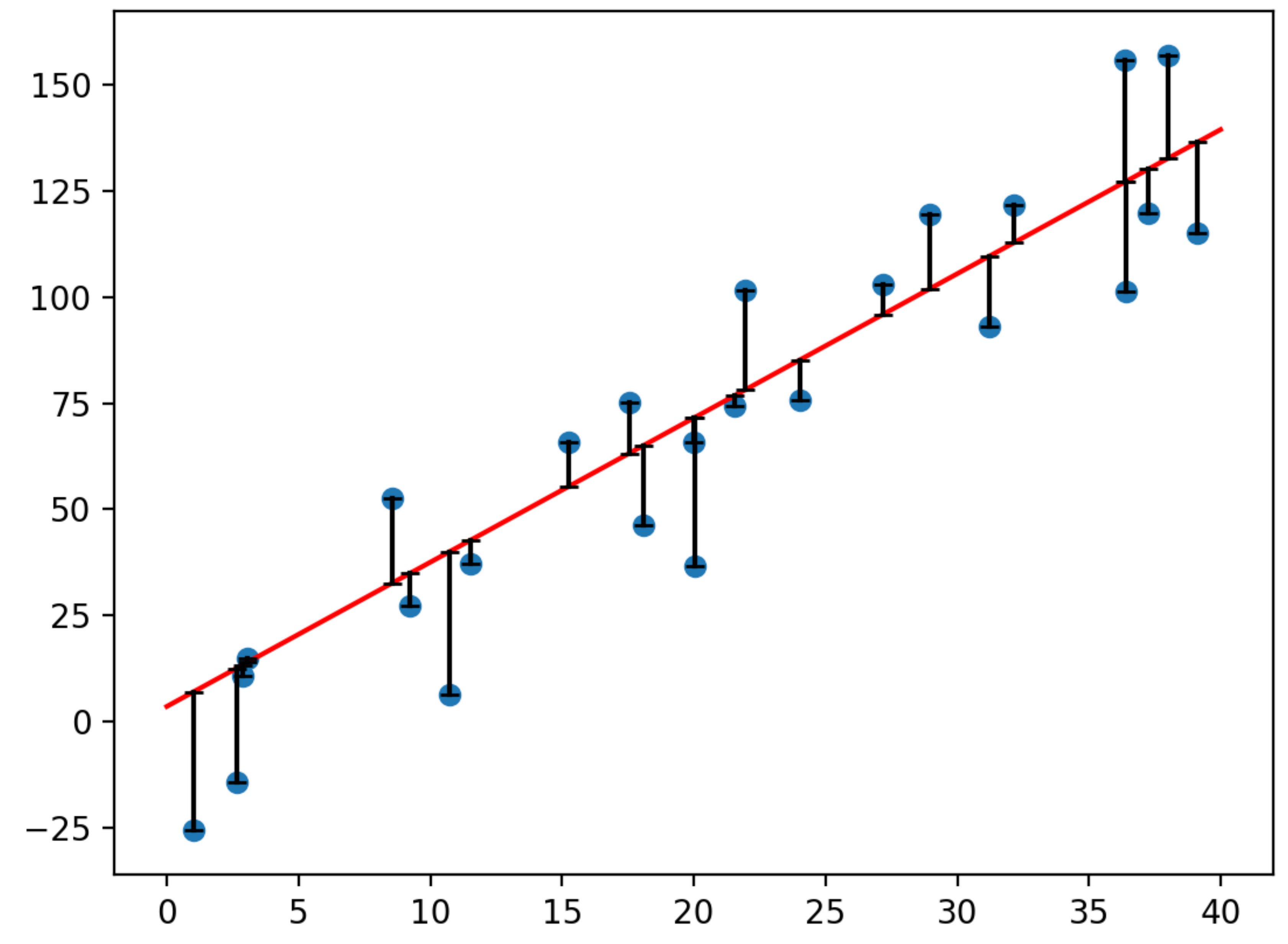


- What do we do if there is more than one explanatory variable?

- To generalize to this case, it is more convenient to work with matrix equations

matrix algebra review

- Let's say $\mathbf{x} = (x_1 \; x_2 \; \cdots \; x_n)^T$ and $\mathbf{y} = (y_1 \; y_2 \; \cdots \; y_n)^T$ are both *n*-dimensional vectors. Then

  $$\mathbf{x}^T\mathbf{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n$$

  is the **inner product** or **dot product** of $\mathbf{x}$ and $\mathbf{y}$, which is the multiplication of a $1 \times n$ and $n \times 1$ vector and results in a scalar.

- For example, suppose $\mathbf{x} = (3 \;\; 4 \;\; 5)^T, \mathbf{y} = (1 \;\; 0 \;\; 2)^T$. Then:

  $$\mathbf{x}^T\mathbf{y} = (3 \quad 4 \quad 5)\begin{pmatrix}1\\0\\2\end{pmatrix} = 3 \times 1 + 4 \times 0 + 5 \times 2 = 13$$

- The **L2-norm** of a vector $\mathbf{x} = (x_1 \; x_2 \; \cdots \; x_n)^T$ is a generalization of the Pythagorean theorem for finding the "length":

  $$\|\mathbf{x}\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

matrix algebra review

- More generally, define two $m \times n$ matrices:

$$\mathbf{X} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m1} & y_{m2} & \cdots & y_{mn} \end{bmatrix}$$

Then the matrix multiplication of $\mathbf{X}^T$ and $\mathbf{Y}$, which results in an $n \times n$ matrix, is:

$$\mathbf{X}^T\mathbf{Y} = [\mathbf{x}_1 \ \mathbf{x}_2 \ \cdots \ \mathbf{x}_n]^T [\mathbf{y}_1 \ \mathbf{y}_2 \ \cdots \ \mathbf{y}_n] = \begin{bmatrix} \mathbf{x}_1^T \\ \mathbf{x}_2^T \\ \vdots \\ \mathbf{x}_n^T \end{bmatrix} [\mathbf{y}_1 \ \mathbf{y}_2 \ \cdots \ \mathbf{y}_n] = \begin{bmatrix} \mathbf{x}_1^T\mathbf{y}_1 & \mathbf{x}_1^T\mathbf{y}_2 & \cdots & \mathbf{x}_1^T\mathbf{y}_n \\ \mathbf{x}_2^T\mathbf{y}_1 & \mathbf{x}_2^T\mathbf{y}_2 & \cdots & \mathbf{x}_2^T\mathbf{y}_n \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{x}_n^T\mathbf{y}_1 & \mathbf{x}_n^T\mathbf{y}_2 & \cdots & \mathbf{x}_n^T\mathbf{y}_n \end{bmatrix}$$

- For example, with $\mathbf{A}$ and $\mathbf{B}$ defined below, we get:

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 2 & 3 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \end{bmatrix} \quad \rightarrow \quad \mathbf{A}^T\mathbf{B} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -2 & -3 \\ 6 & 0 & 2 \\ 10 & 2 & 6 \end{bmatrix}$$

matrix algebra review

- If $\mathbf{X}$ has dimension $a \times b$, and $\mathbf{Y}$ has dimension $c \times d$, then the matrix product $\mathbf{XY}$ is only possible if $b = c$ (i.e., the inner dimensions match), which will have dimension $a \times d$ (outer dimensions)

- If $\mathbf{X}$ is a **square** matrix (i.e., has dimension $n \times n$), then its inverse is $\mathbf{X}^{-1}$ (if it exists), and:

$$\mathbf{X}^{-1}\mathbf{X} = \mathbf{X}\mathbf{X}^{-1} = \mathbf{I}, \text{ where } \mathbf{I} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

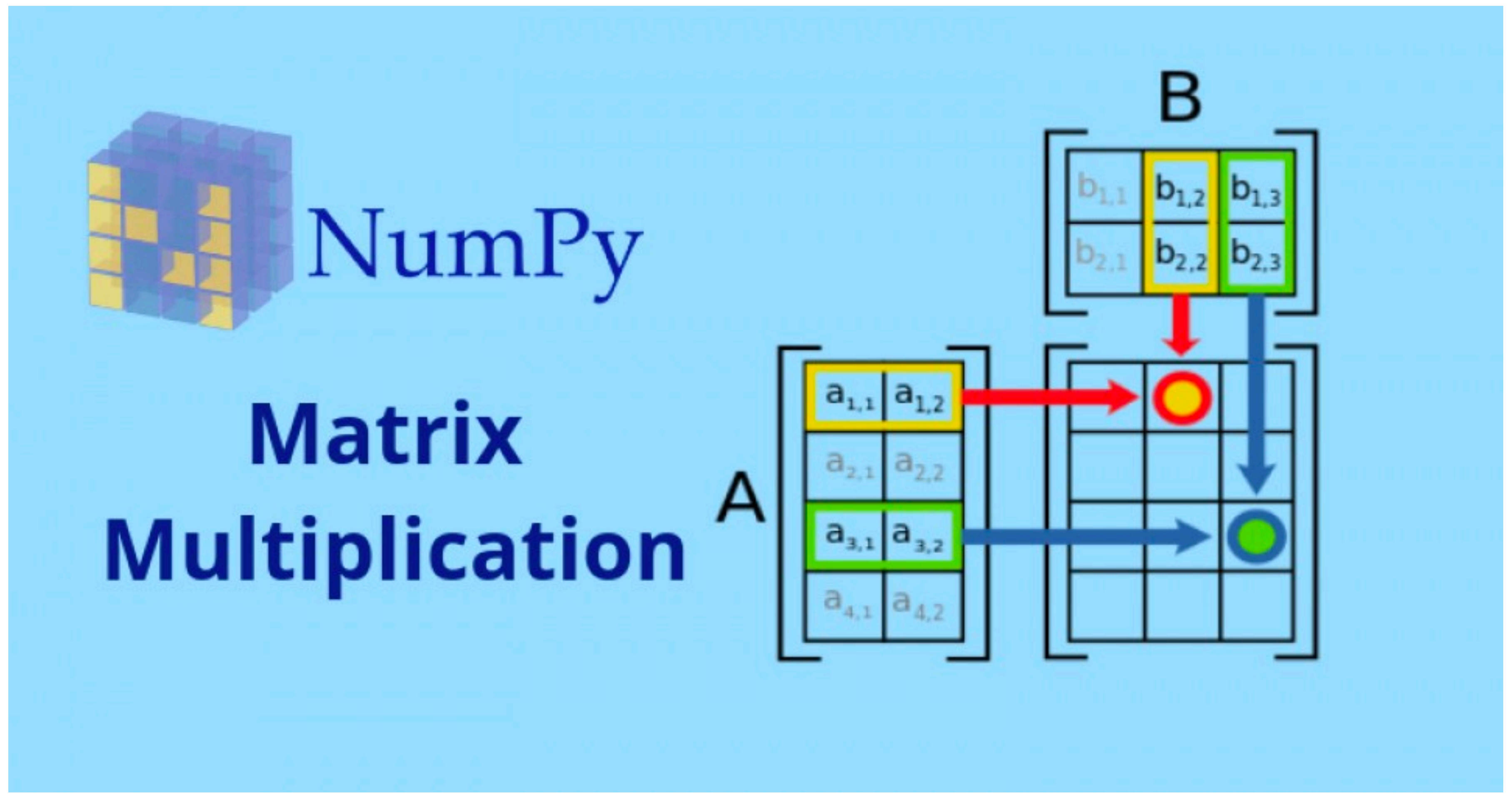
  is the $n \times n$ **identity matrix**

- For example, with $\mathbf{A}$ and $\mathbf{B}$ defined as below, we can verify $\mathbf{B} = \mathbf{A}^{-1}$, since $\mathbf{AB} = \mathbf{I}$:

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0.2 & 0.2 & 0 \\ -0.2 & 0.3 & 1 \\ 0.2 & -0.3 & 0 \end{bmatrix}, \quad \mathbf{AB} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

numpy

- But how do we perform matrix manipulations, like taking inverses, on large matrices in general?
- In Python, we can use the numpy library to do matrix operations



```
import numpy as np

np.array(A)   //Convert list to numpy array

np.matmul(A,B)  //Matrix multiplication (or A@B)

np.linalg.inv(A)  //Matrix inverse

A.sum(axis=0)  //Sum over rows of matrix
```

- See https://scipy-lectures.org/intro/numpy/operations.html for more examples, as well as the notebook

matrix form of linear regression equations

- Now, back to regression
- For **simple linear regression**, if we define

$$\mathbf{X} = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_N & 1 \end{bmatrix} \qquad \beta = \begin{bmatrix} a \\ b \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}$$

then we can write the equations for all data points compactly using the following matrix equation:

$$\mathbf{y} = \mathbf{X}\beta + \epsilon$$

- The **multivariable linear regression model** with M explanatory variables is

$$y_n = a_1 x_{n,1} + a_2 x_{n,2} + \cdots + a_M x_{n,M} + b + \epsilon_n, \quad n = 1,\ldots,N$$

- In this case, we define

$$\mathbf{X} = \begin{bmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,M} & 1 \\ x_{2,1} & x_{2,2} & \cdots & x_{2,M} & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{N,1} & x_{N,2} & \cdots & x_{N,M} & 1 \end{bmatrix} \qquad \beta = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_M \\ b \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}$$

where $\mathbf{X}$ is the **feature matrix**. Then, as before, we can write

$$\mathbf{y} = \mathbf{X}\beta + \epsilon$$

least squares equations

- With this matrix notation, we can write our original optimization for minimizing MSE as:

$$\min_{\beta} \frac{1}{N} \sum_{n=1}^{N} (y_n - \mathbf{x}_n^T \beta)^2$$

- Or, equivalently, this can be written using the vector norm:

$$\min_{\beta} \frac{1}{N} \|\mathbf{y} - \mathbf{X}\beta\|_2^2$$

- Similar to 1D case, we can take the **gradient** (multidimensional derivative) and set to **0** (i.e., the vector of zeros) to find minimum:

$$\nabla((1/N)\|\mathbf{y} - \mathbf{X}\beta\|_2^2) = (2/N)\mathbf{X}^T\mathbf{X}\beta - (2/N)\mathbf{X}^T\mathbf{y} = \mathbf{0}$$

- This yields the **least squares equations** for solving for β:

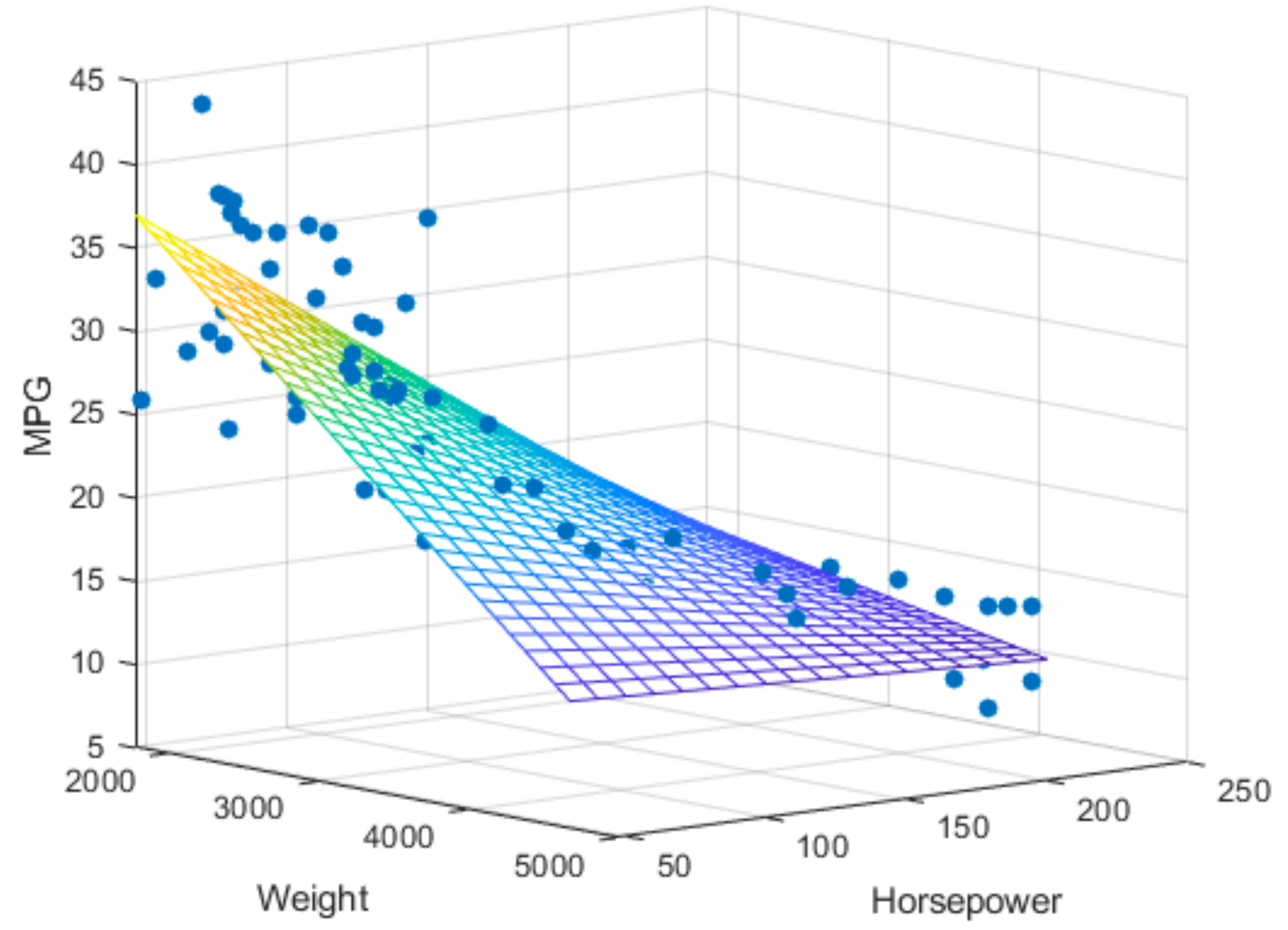
$$\mathbf{X}^T\mathbf{X}\beta = \mathbf{X}^T\mathbf{y}$$

solving for β

- If $\mathbf{X}^T\mathbf{X}$ is invertible, we can take a matrix inverse to solve for the model parameters β:

$$\beta = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}$$

- But $\mathbf{X}^T\mathbf{X}$ is not always invertible
    - The inverse exists if and only if the columns of $\mathbf{X}$ are **linearly independent** of one another
    - This means that we cannot have the case where one column can be written as a linear combination of the others
- What does it mean when $\mathbf{X}^T\mathbf{X}$ is not invertible?
    - Infinitely many possible solutions
    - We typically choose the one where $||\beta||$ is smallest. Why?



example

Suppose we collect five data points consisting of two features x_1, x_2 and a target variable y in the form (x_1, x_2, y): (1, 2, 10), (-3, 6, 0), (0, 0, 3), (1, -1, 4), (5, -2, 20). We want to fit a linear regression model to this dataset.

Formulate the least squares equations. What is the resulting model? What would be the prediction for a new datapoint with $x_1 = -1, x_2 = 1$?

solution

The model we want to fit is $\hat{y} = a_1x_1 + a_2x_2 + b$, where $\beta = (a_1 \ a_2 \ b)^T$ is the parameter vector. The feature matrix $\mathbf{X}$, target vector $\mathbf{y}$, and least squares equations are:

$$\mathbf{X} = \begin{bmatrix} 1 & 2 & 1 \\ -3 & 6 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \\ 5 & -2 & 1 \end{bmatrix}, \qquad \mathbf{y} = \begin{pmatrix} 10 \\ 0 \\ 3 \\ 4 \\ 20 \end{pmatrix},$$

$$\begin{bmatrix} 1 & -3 & 0 & 1 & 5 \\ 2 & 6 & 0 & -1 & -2 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ -3 & 6 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \\ 5 & -2 & 1 \end{bmatrix} \beta = \begin{bmatrix} 1 & -3 & 0 & 1 & 5 \\ 2 & 6 & 0 & -1 & -2 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{pmatrix} 10 \\ 0 \\ 3 \\ 4 \\ 20 \end{pmatrix}$$

solution

Using the `numpy` commands for inverse, transpose, and multiplication, we get

$$\beta = (4.2308,\ 1.7538,\ 2.2615)^T$$

Which means that our model is

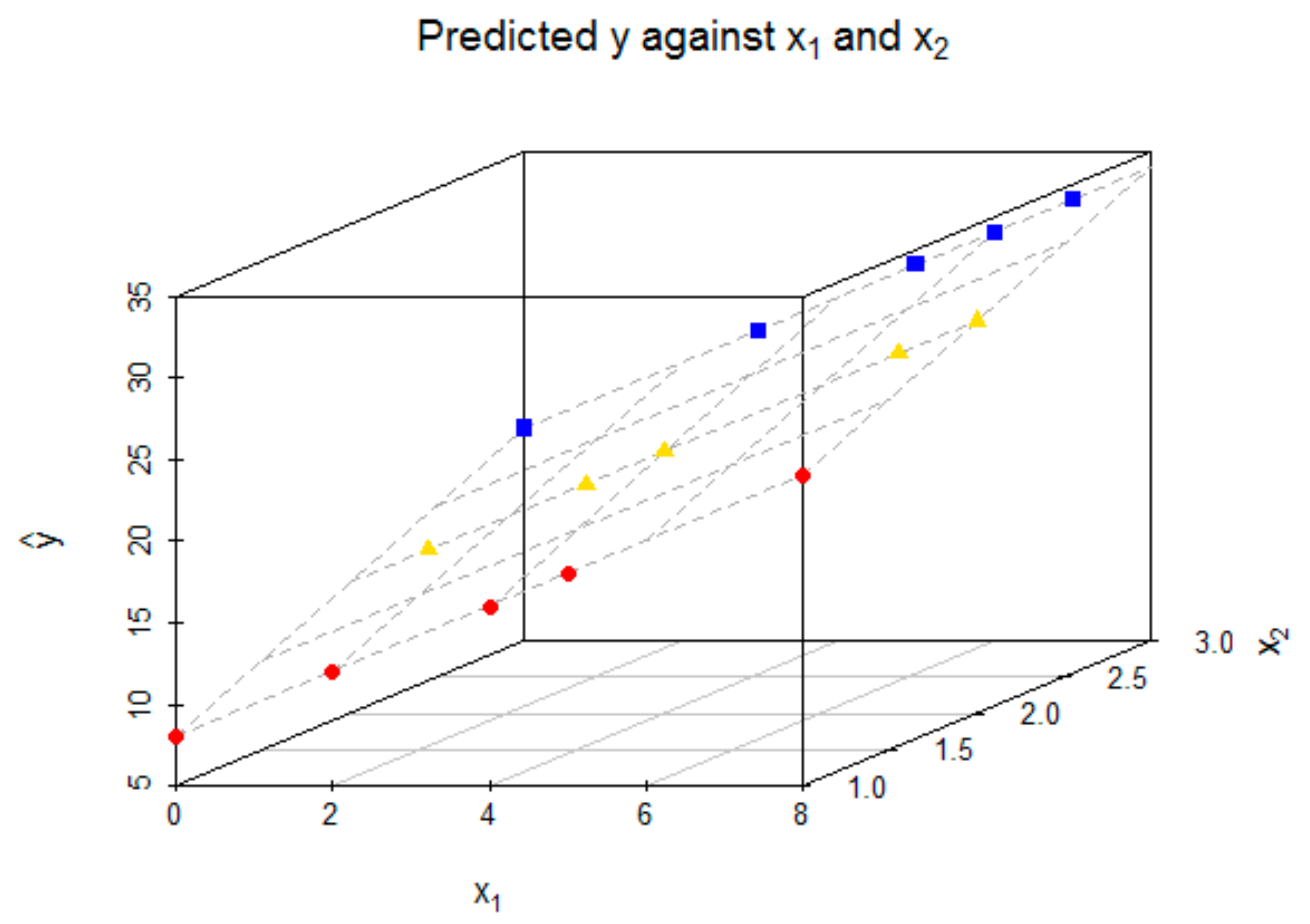
$$\hat{y} = 4.2308x_1 + 1.7538x_2 + 2.2615$$

And the prediction for $x_1 = -1, x_2 = 1$ is

$$\hat{y} = 4.2308 \cdot -1 + 1.7538 \cdot 1 + 2.2615 = -0.2154$$

interpreting results

- How should we interpret the results of linear regression?
  - Recall multi-feature model, e.g., $y_n = a_1 x_{n,1} + a_2 x_{n,2} + b$
  - If one feature weight (e.g., a_1) is higher than another (e.g., a_2), this can indicate that this feature is more important than the other (contributes more to the value of y)
- Need to be careful, though! If different features have different scales, then weights will naturally be different!
  - Normalization is useful as it standardizes the feature ranges



Here, x_1 has a range of 8, while x_2 only has a range of 2

normalization for interpretation

- *Problem:* Suppose I fit a linear regression model and get

  $$\hat{y} = 10x_1 + 100x_2 + 5$$

  - Does this mean that x_2 has a bigger impact on y than x_1?
  - Not necessarily, because we have said nothing about the ranges of x_1 and x_2 that resulted in $a_1 = 10$ and $a_2 = 100$.

- *One solution:* **Normalize** the data before doing linear regression so that coefficients are compared over a consistent range.

standard normalization

- For every feature column, do the following to make them all have a mean of 0 and standard deviation of 1:

  1. *Center values*: Subtract the column average from each feature sample
     - Useful to eliminate any bias contained in the features
  2. *Scale values*: Divide each feature sample by the column standard deviation
     - Re-scales features so that each is expressed in new units: standard deviations from the mean (similar to how we calculate z-scores)

- Mathematically, we are defining the following operation for each feature column $\mathbf{x}_m$:

  $\tilde{\mathbf{x}}_m = \dfrac{\mathbf{x}_m - \bar{x}_m}{s_m}$, where $\bar{x}_m$ and s_m are the sample mean and standard deviation of feature m

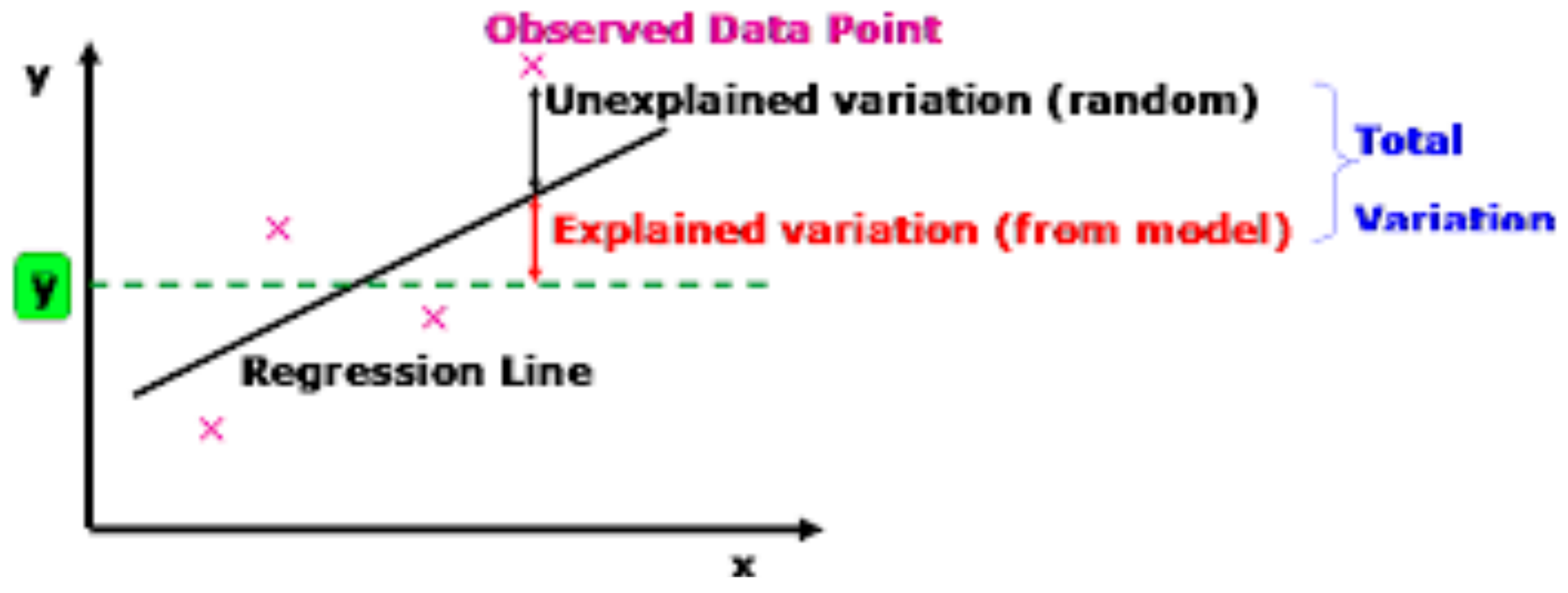
coefficient of determination

- How good is the fit of the regression to the dataset?
- To answer this, one possibility is using the MSE
- Another commonly used quantity is the **coefficient of determination**, called r^2

$$r^2 = 1 - \frac{\sum_{n=1}^{N} (y_n - \hat{y}_n)^2}{\sum_{n=1}^{N} (y_n - \bar{y})^2} = 1 - \frac{MSE}{\sigma_Y^2}$$

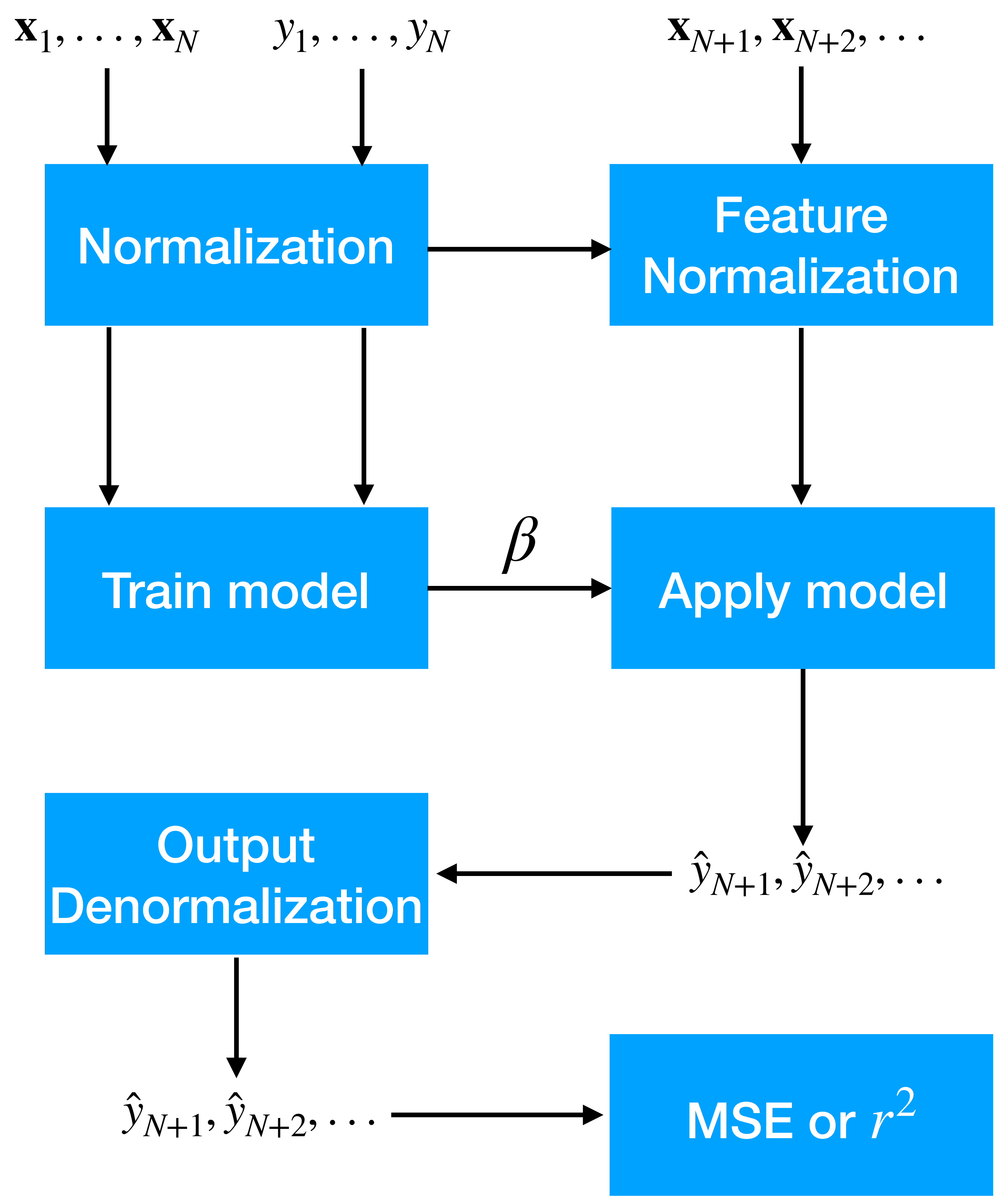
- y_n: Measured value, $\hat{y}_n$: Predicted value
- $\bar{y}$: Mean measured value, σ_Y^2: Variance of measured value

- r^2 gives the fraction of variance in the data that is explained by the model
- Typically between 0 (bad, no better than horizontal line) and 1 (perfect fit)
  - Sometimes preferred to MSE in regression problems for this reason


Observed Data Point
Unexplained variation (random)
Total Variation
Explained variation (from model)
y
ȳ
Regression Line
x

using your model after fitting

- After fitting a linear regression model, you can **estimate** (or predict) the target y of new datapoints using your model
  - New data point: $(x_1, x_2, \ldots)$
  - Prediction: $\hat{y} = a_1 x_1 + a_2 x_2 + \cdots + b$
- How good is the prediction?
  - Squared error between $\hat{y}$ and y (once it is known)
  - MSE or r^2 over a set of new data points
- When using the model, make sure to take into account any normalization that was used (i.e., normalize new datapoints before inputting them, "un-normalize" the $\hat{y}$ you get back)


$\mathbf{x}_1, \ldots, \mathbf{x}_N$
$y_1, \ldots, y_N$
$\mathbf{x}_{N+1}, \mathbf{x}_{N+2}, \ldots$
Normalization
Feature Normalization
Train model
β
Apply model
Output Denormalization
$\hat{y}_{N+1}, \hat{y}_{N+2}, \ldots$
$\hat{y}_{N+1}, \hat{y}_{N+2}, \ldots$
MSE or r^2

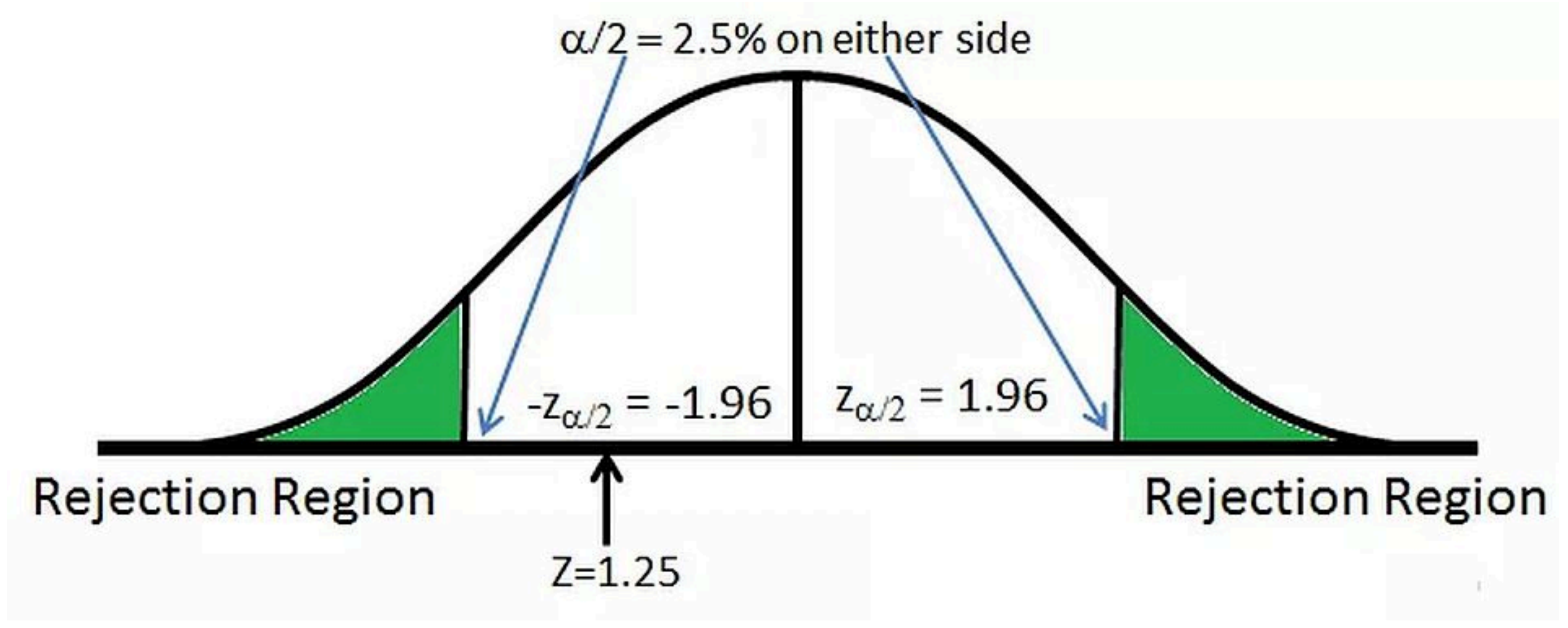
linear regression in python

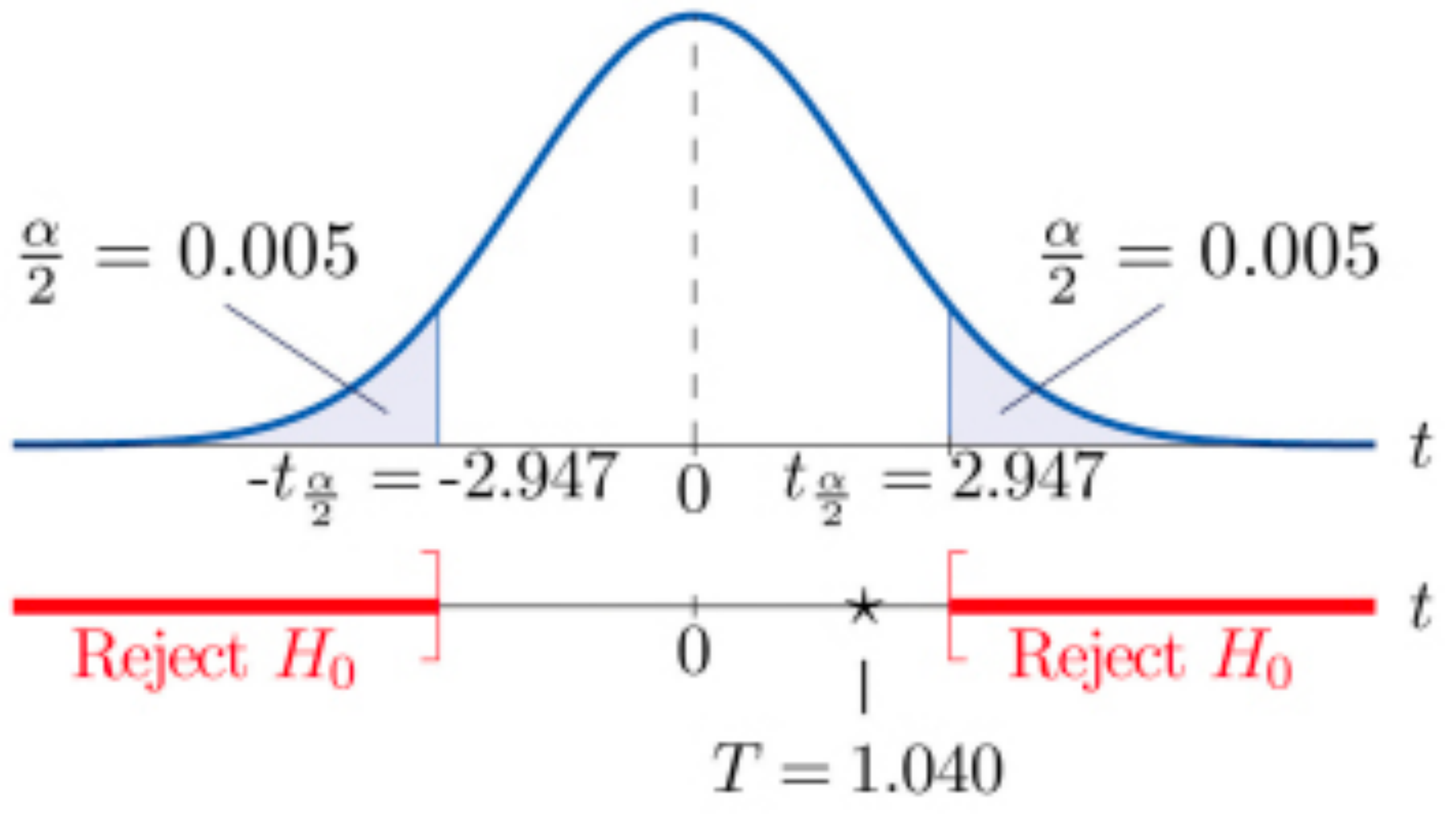
- You can solve the least squares equations directly using `numpy`
- Given how common linear regression is, several variants are built in to the `sklearn` (`scikit learn`) library directly:

```
from sklearn import linear_model, from sklearn.metrics import
mean_squared_error, r2_score

regr = linear_model.LinearRegression(fit_intercept=True)  # Define
linear regression object

regr.fit(X_train,y_train)  # Fit model to training set

regr.coef_  # View coefficients (a_1,…,a_M) of trained model

regr.intercept_  # View intercept (b) of trained model

y_pred = regr.predict(X_test)   # Apply model to test set

r2_score(y_true,y_pred)  # r2 score between true and predicted
```

more interpretation

- Is a feature significant?
  - Just because a feature is used in a model doesn't mean it is important in predicting the value of the output
  - But the model will try to account for the feature anyway!
- Can perform a hypothesis test (see previous lectures):
  - *Null hypothesis* H_0: Coefficient a_m is 0 (feature has no predictivity, y does not depend on x_m)
  - *Alternative hypothesis* H_1: Coefficient a_m is not 0 (feature has predictivity, y does depend on x_m)

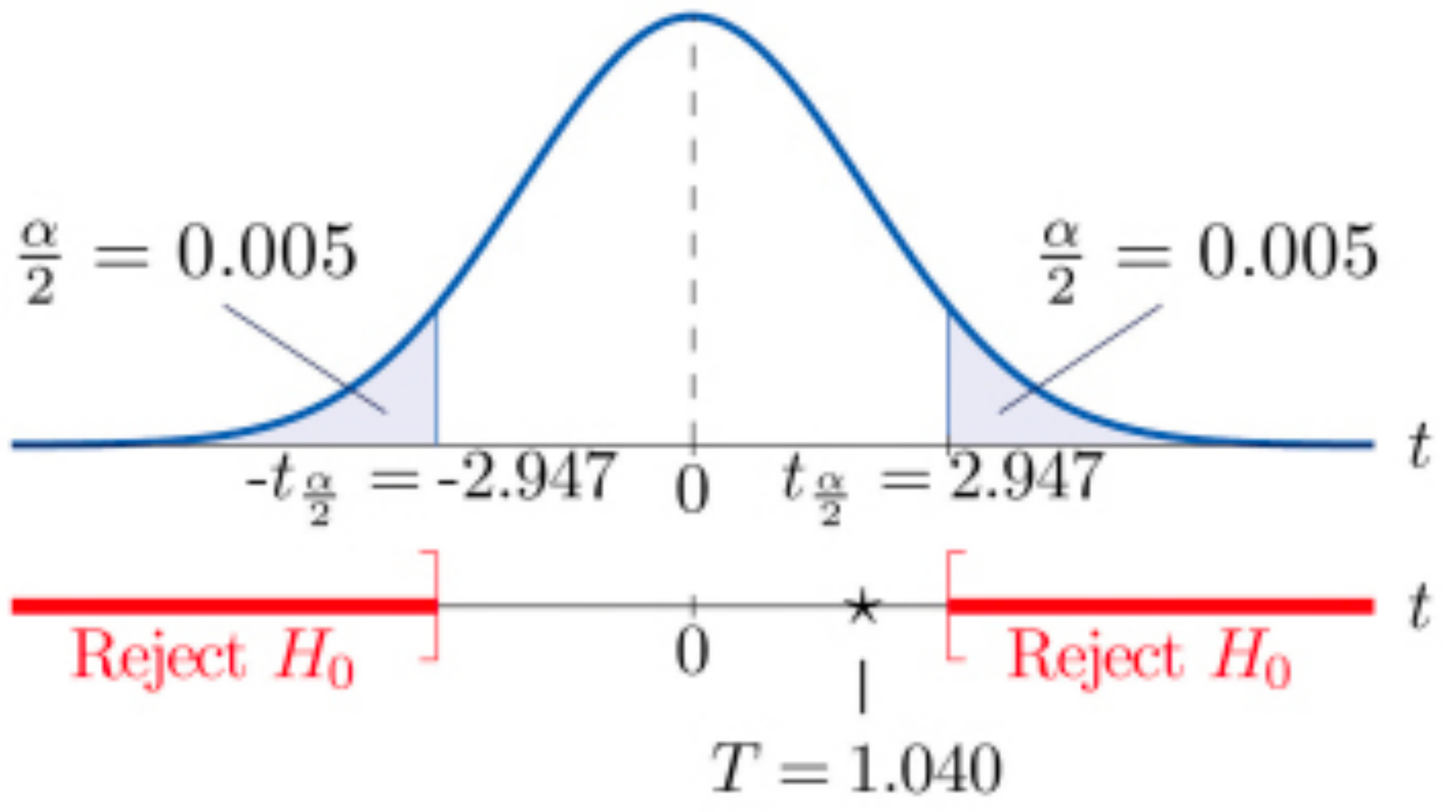



$\frac{\alpha}{2} = 0.005$
$\frac{\alpha}{2} = 0.005$
$-t_{\frac{\alpha}{2}} = -2.947$
0
$t_{\frac{\alpha}{2}} = 2.947$
t
Reject H_0
0
Reject H_0
t
$T = 1.040$

hypothesis test for regression

- Test statistic is always: (value - hypothesized value) / standard error

$$\frac{\hat{a}_m - a_m}{SE_{a_m}} \Rightarrow \frac{\hat{a}_m}{SE_{a_m}}$$



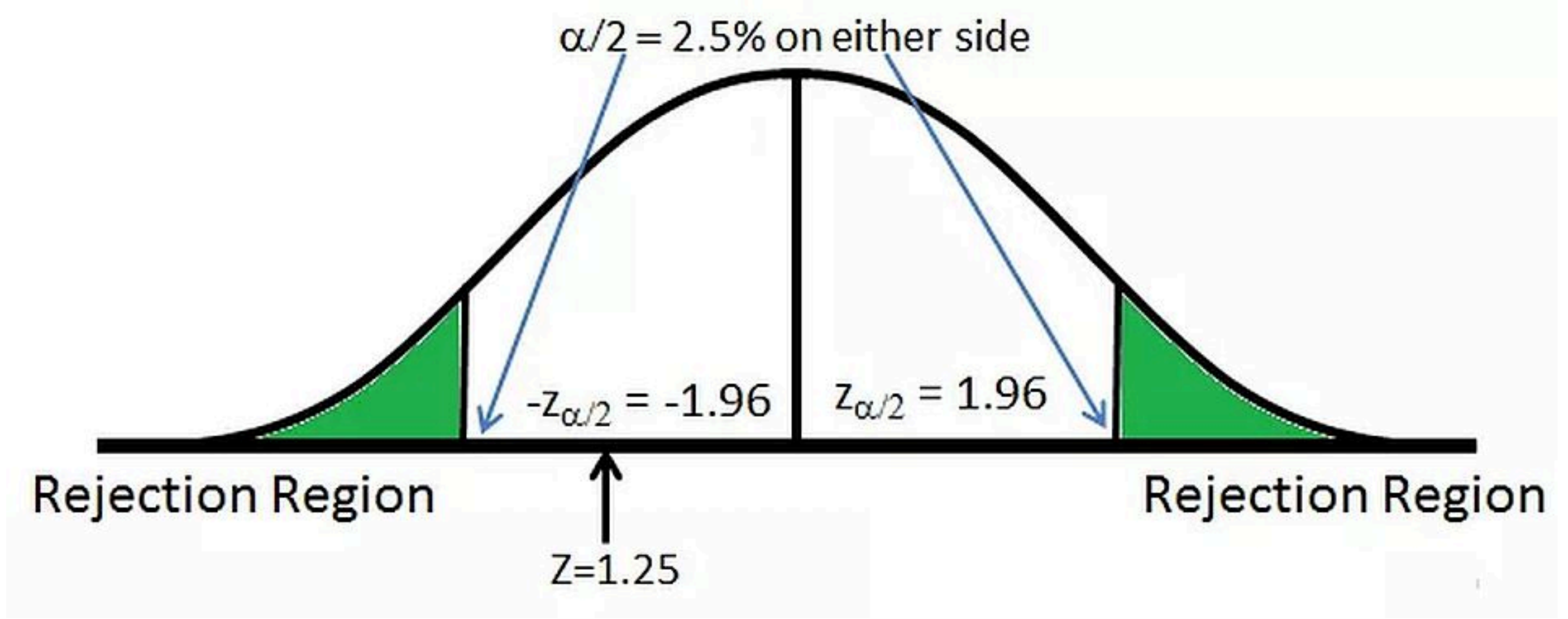
- What is the standard error for a regression coefficient a_m?

$$SE_{a_m} = \frac{\sqrt{\frac{\sum_{n=1}^{N} (y_n - \hat{y}_n)^2}{N-2}}}{\sqrt{\sum_{n=1}^{N} (x_{n,m} - \bar{x}_m)^2}}$$

- y_n: Measured value, $x_{n,m}$: Feature value
- $\hat{y}_n$: Predicted value, $\bar{x}_m$: Feature average

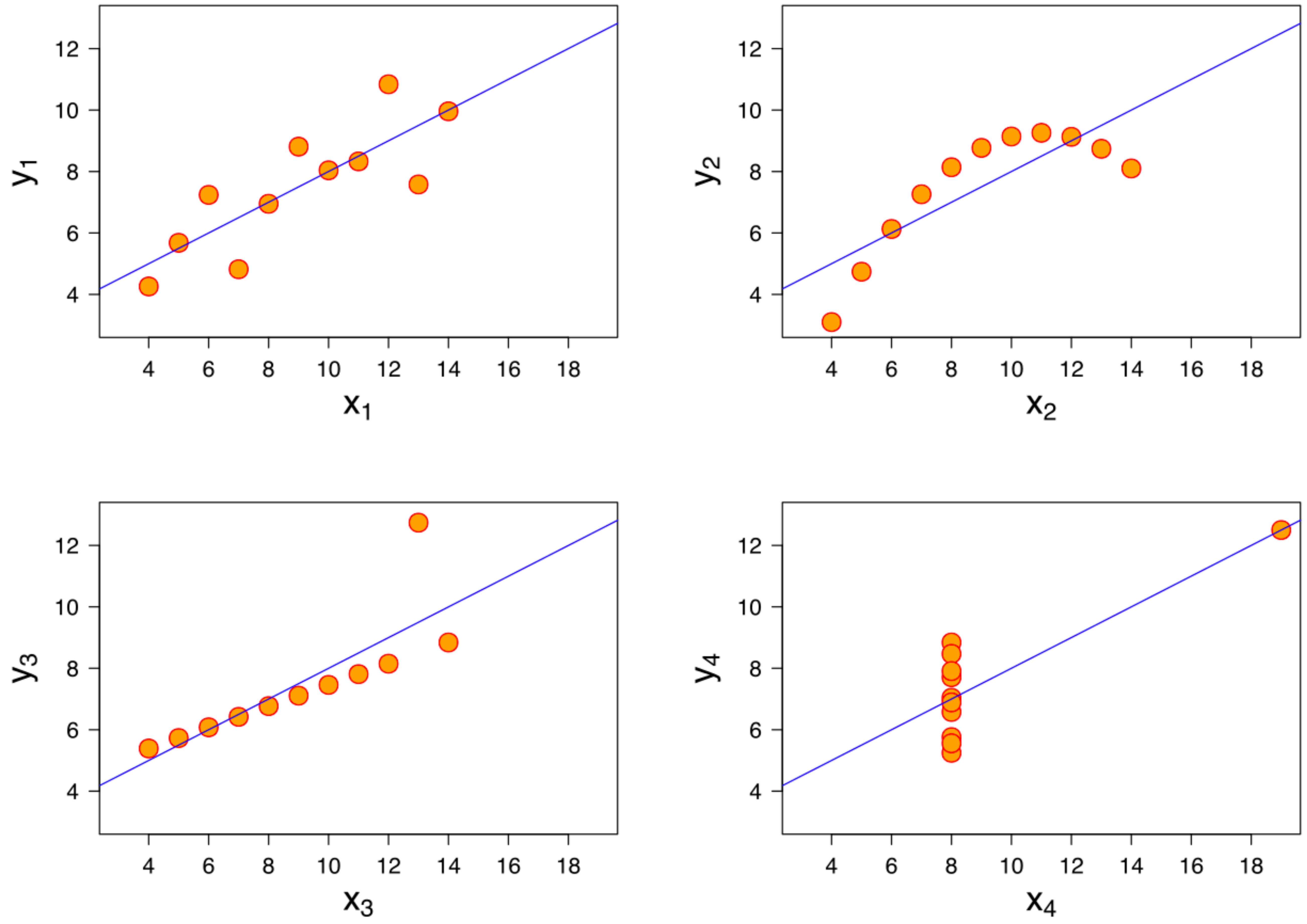
- For a z-test, find p-value of SE_{a_m} against the z-distribution

- For a t-test, find p-value against a t-distribution with $N - k - 1$ degrees of freedom, where k is the number of features



a linear model may be wrong

- In these graphs, all 4 datasets have the same …
  - linear regression line
  - coefficient of determination
  - mean and variance of both x and y
- Yet clearly, the relationship between x and y is different in each case
- It is important to visualize the results, and possibly try non-linear models!



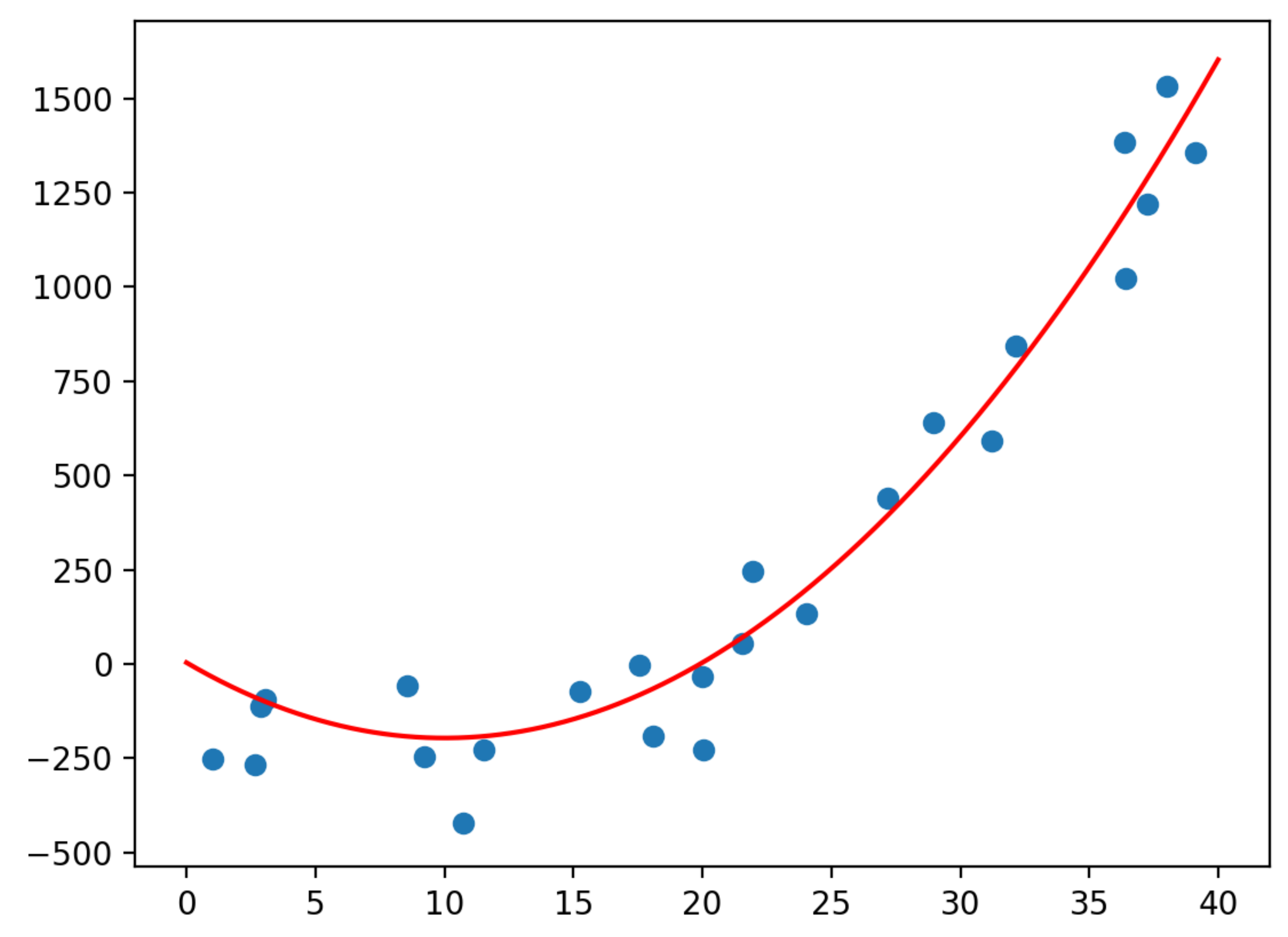
what about non-linear?

- A common (and understandable) misconception is that linear regression can only find linear relationships
  - The “linear” part refers to the parameter vector β, not the input features in $\mathbf{X}$
- We can readily take nonlinear functions of our features
- For example, suppose we want to fit a quadratic model:

$$y_n = a_1(x_n)^2 + a_2 x_n + b$$

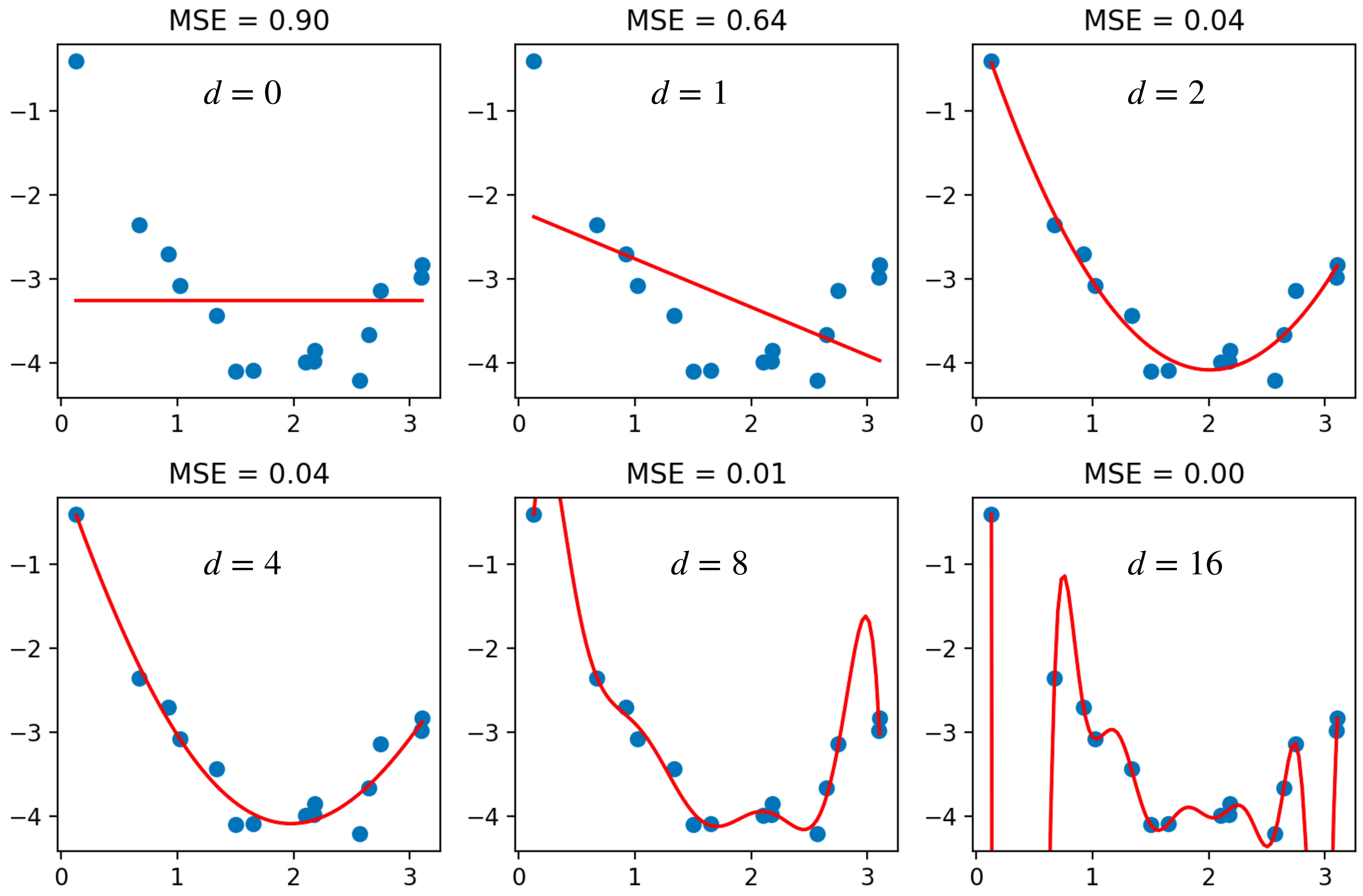
- We create a “synthesized” feature matrix that has the quadratic form:

$$\mathbf{X} = \begin{bmatrix} (x_1)^2 & x_1 & 1 \\ (x_2)^2 & x_2 & 1 \\ \vdots & \vdots & \vdots \\ (x_N)^2 & x_N & 1 \end{bmatrix} \quad \beta = \begin{bmatrix} a_1 \\ a_2 \\ b \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}$$



more and more complexity

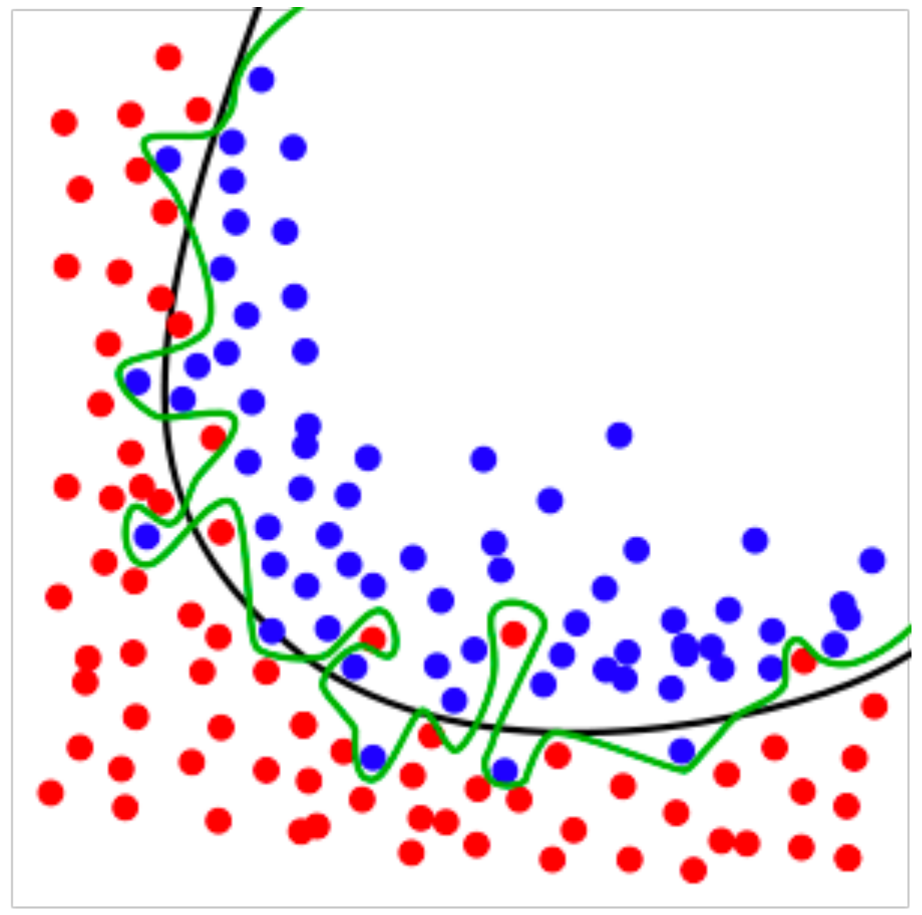
- If we use a higher degree d of polynomials, we can reduce MSE:

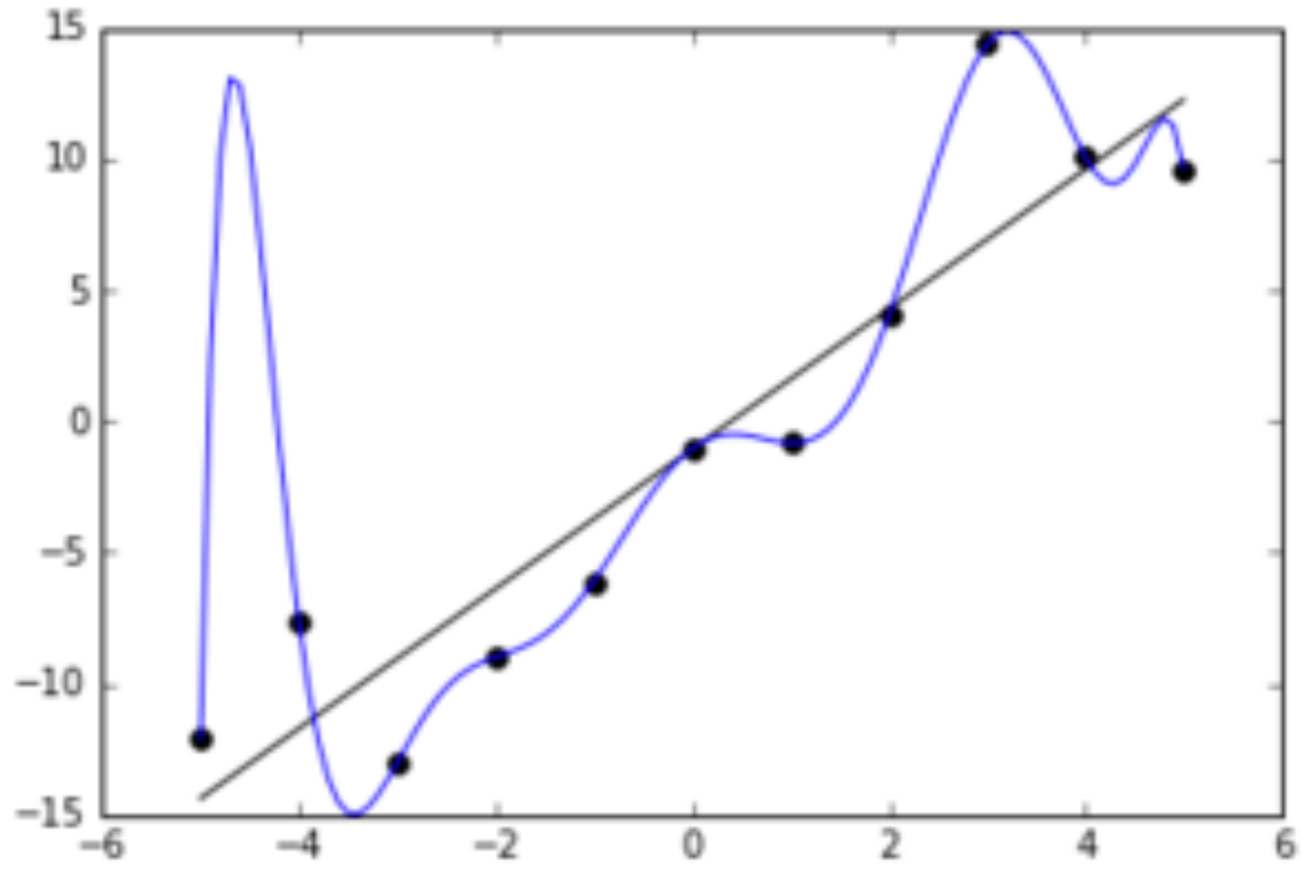


- But, is this a good thing to do?

overfitting

- If our goal was just to minimize error on the existing dataset, we'd keep adding features (e.g., increasing the degree d of a polynomial)
- But this sacrifices the generalizability of the model
- An **overfitted** model is one which contains too many parameters than can be justified by the data
  - High r^2 and low MSE on training data, but low r^2 and high MSE on testing data
- We can contrast this with **underfitting**, where we don't have enough parameters to drive down MSE on either training or testing data



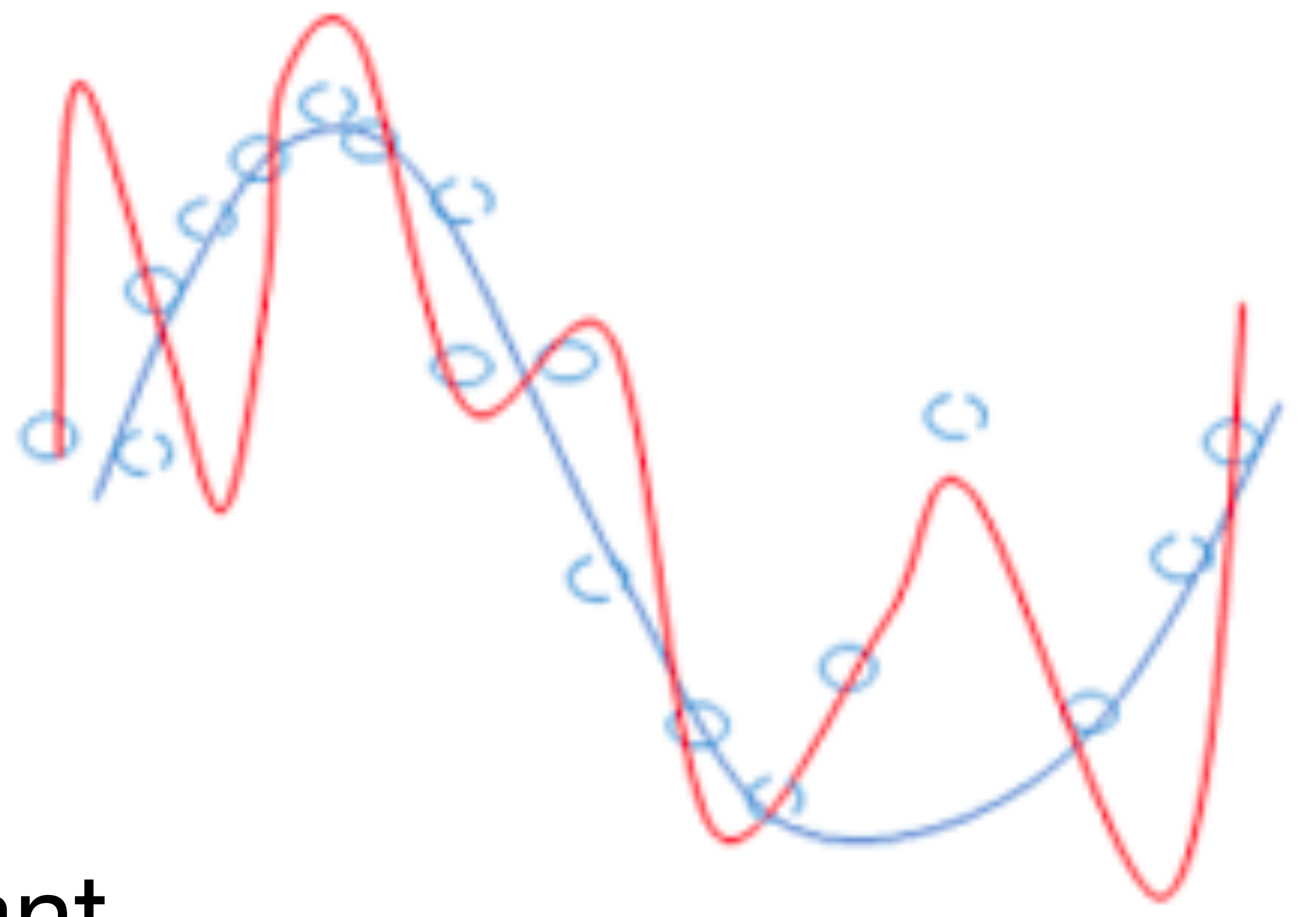


regularization

- When we have a lot of features, we can use **regularization**, a class of techniques for mitigating overfitting by penalizing non-zero model coefficients
- The general expression we work with in regularization is:

```
    minimize (model error) + λ(coefficient weights)
```

- $\lambda \geq 0$ is the **regularization parameter**
  - Higher λ: Minimizing model parameters becomes more important
  - Lower λ: Minimizing model error becomes more important
- Several different regularization techniques: Lasso, **Ridge**, Elastic-Net, …



ridge regression

- In **ridge regression**, the regularization term is the sum of squares of the coefficients:

$$\underset{\beta}{\text{minimize}}\ ||\mathbf{X}\beta - \mathbf{y}||_2^2 + \lambda||\beta||_2^2$$

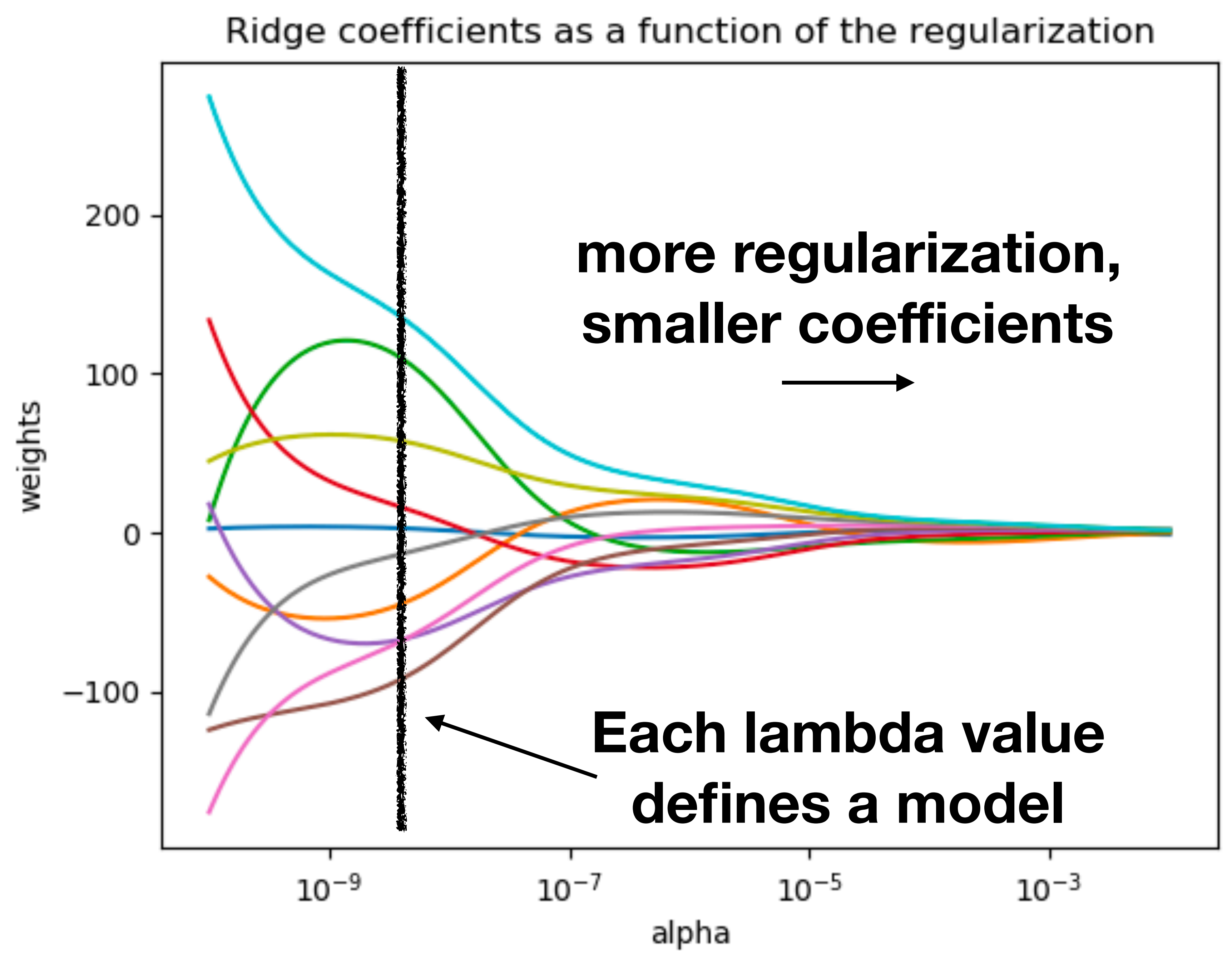
- This makes it easy to solve in matrix form as:

$$\beta^\star = (\mathbf{X}^T\mathbf{X} + \lambda\mathbf{I})^{-1}\mathbf{X}^T\mathbf{y}$$

- In Python:

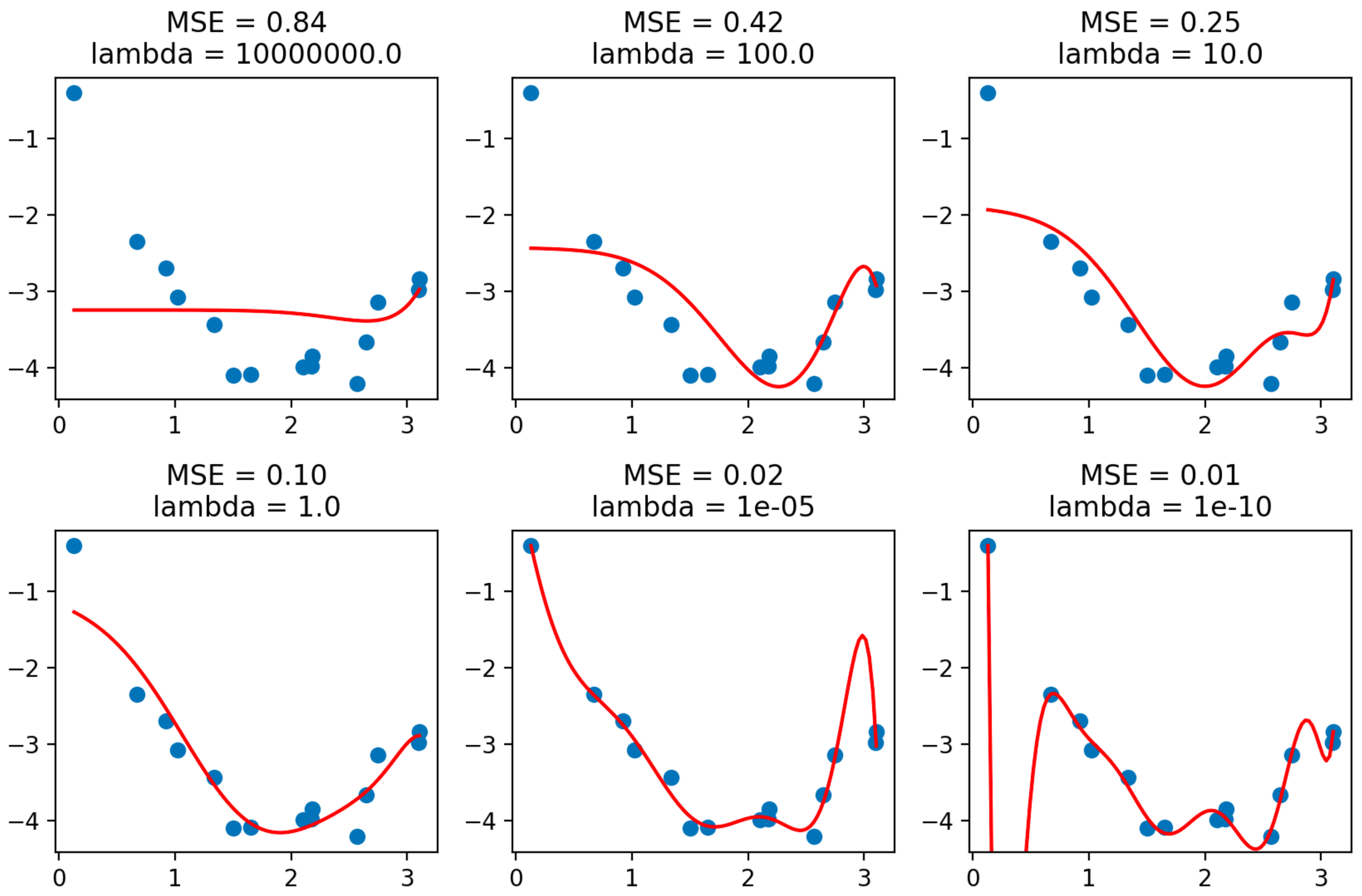
```
from sklearn import linear_model

reg = linear_model.Ridge(lambda=0.1, fit_intercept=True)
```

regularization parameter



regularization can alleviate overfitting

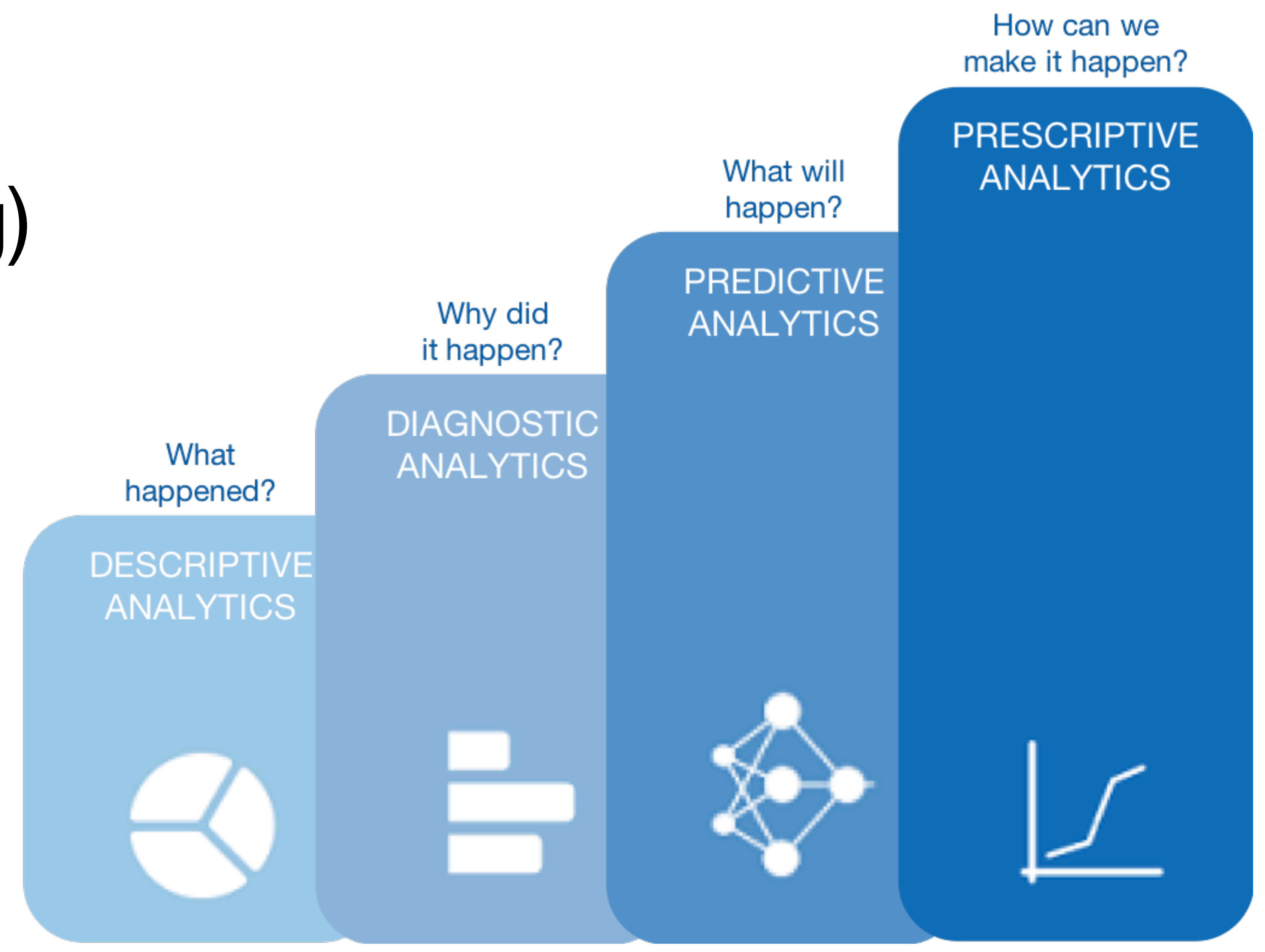
- Polynomial of degree $d = 10$, with different amounts of regularization:



- A higher value of λ has a “smoothing” effect on the model

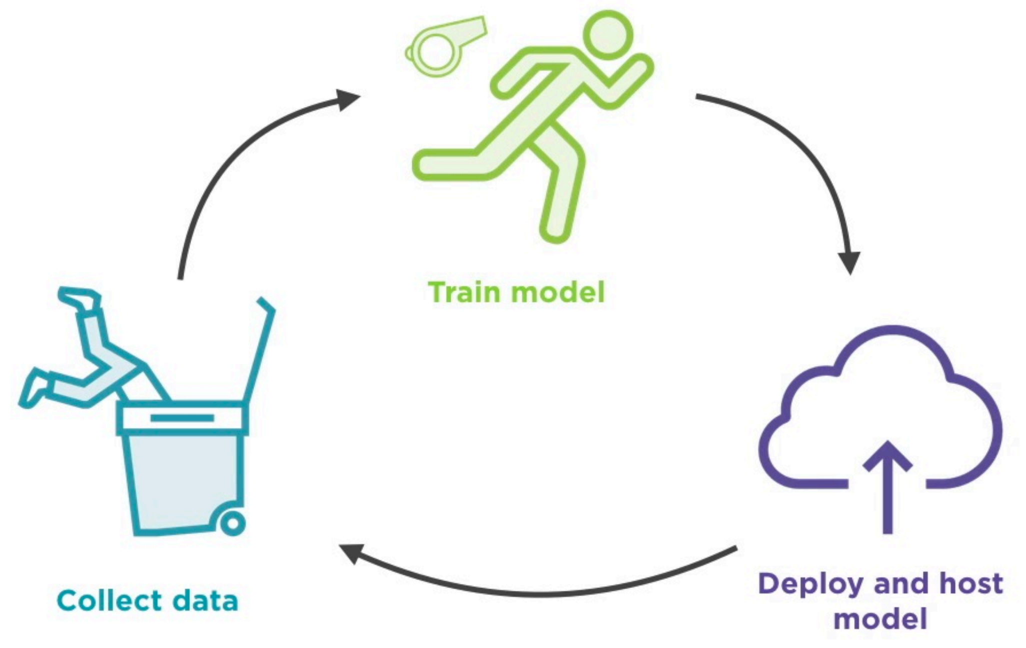
evaluating predictive performance

- Descriptive and diagnostic analysis (classical statistics, data mining)
  - Focus: Understand and interpret statistical relationships in *observed dataset*
  - Evaluation: e.g., MSE or r^2 on **training data** (data used to fit the model)
- Predictive and prescriptive analysis (machine learning)
  - Focus: Predict target value for *new or future unseen data*
  - Evaluation: e.g., MSE or r^2 on **test data** (data <u>not</u> used to fit the model)



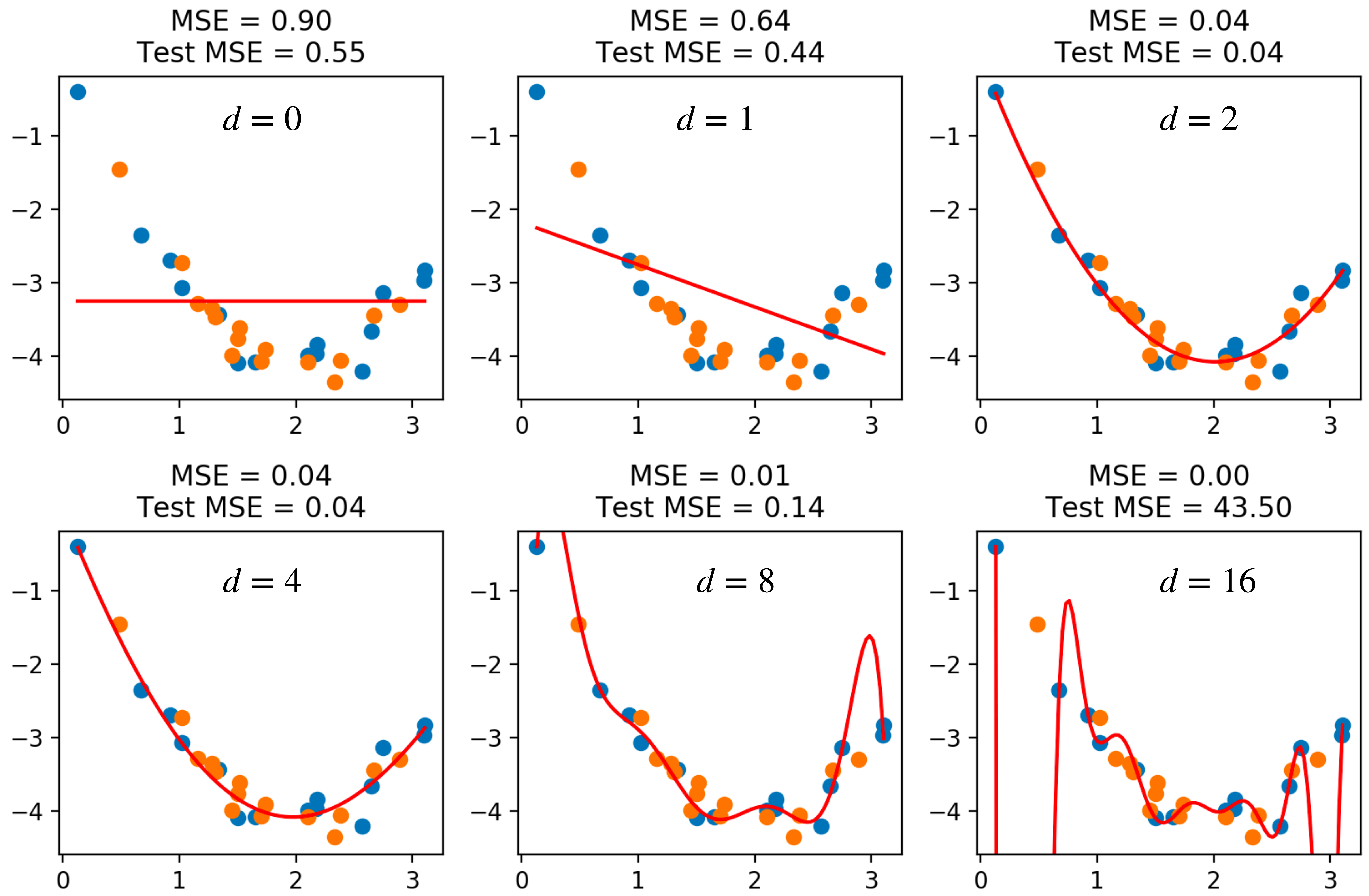
why evaluate on test data?

- Analogy to class
  - **Training data** is like homeworks, sample problems and sample exams
  - **Testing data** is like the real exam
- If we train and evaluate on the same data, the model may not generalize well
- Reasons for computing performance on *test data* (the standard ML approach):
  - **Model evaluation**: Quantify the model's predictive performance *if deployed*
    - e.g., describing the model and its business implications to the CEO
  - **Model selection**: Select which model should be deployed
    - e.g., which polynomial degree or regularization value should be used?



choosing model based on test MSE

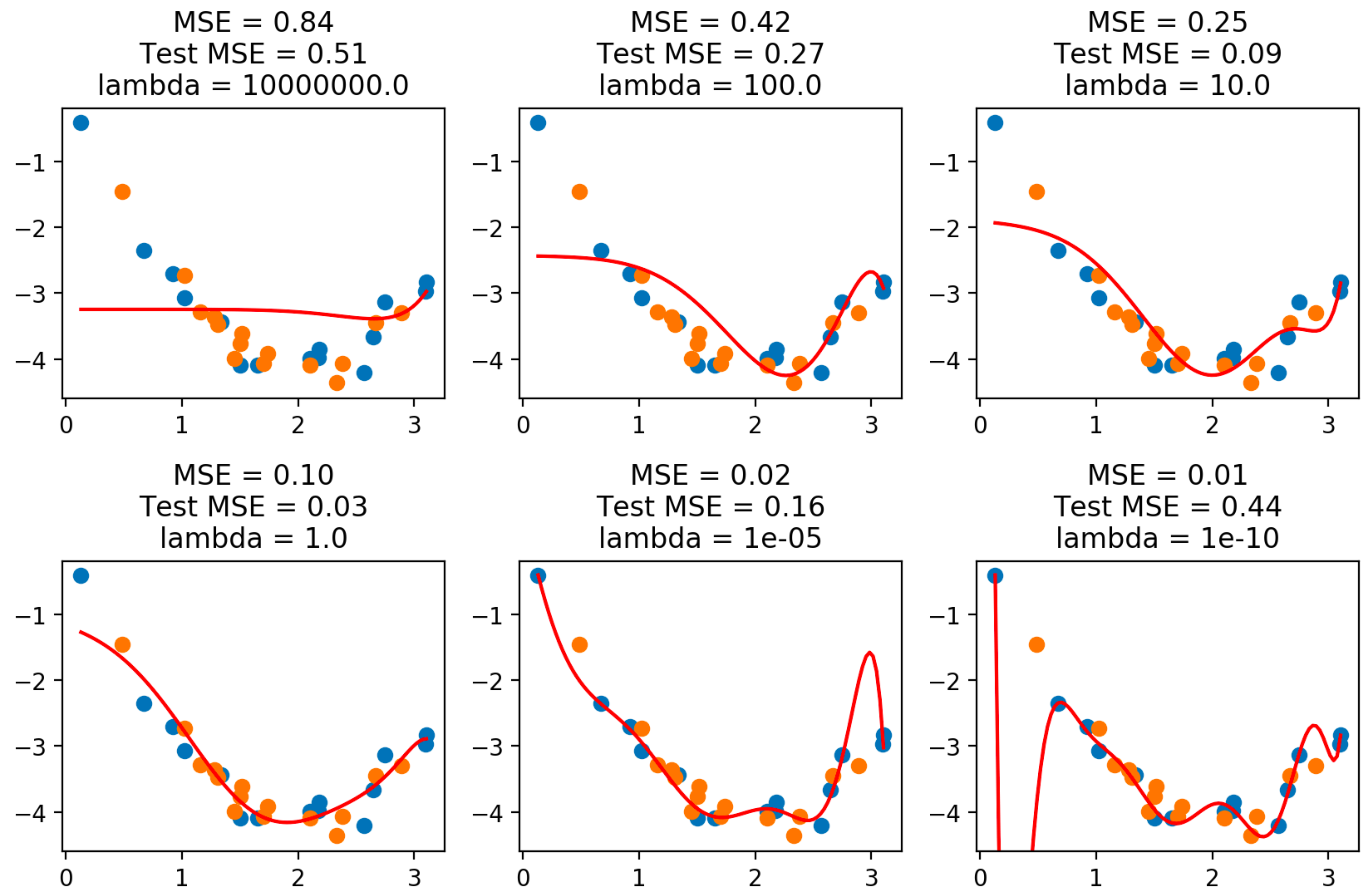
- We can use MSE on a held-out test set to determine the best model:



- Blue points: Training set
- Orange points: Held-out test set

choosing model based on test MSE

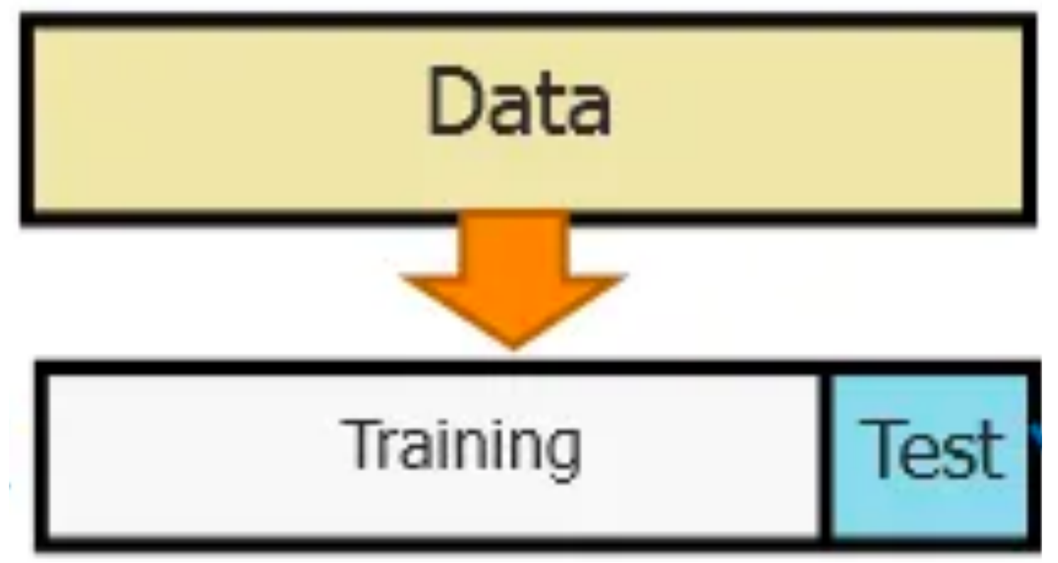
- We can use MSE on a held-out test set to determine the best model:



- The best model has the lowest test MSE
- This is often achieved when there is a small difference between training and test MSE

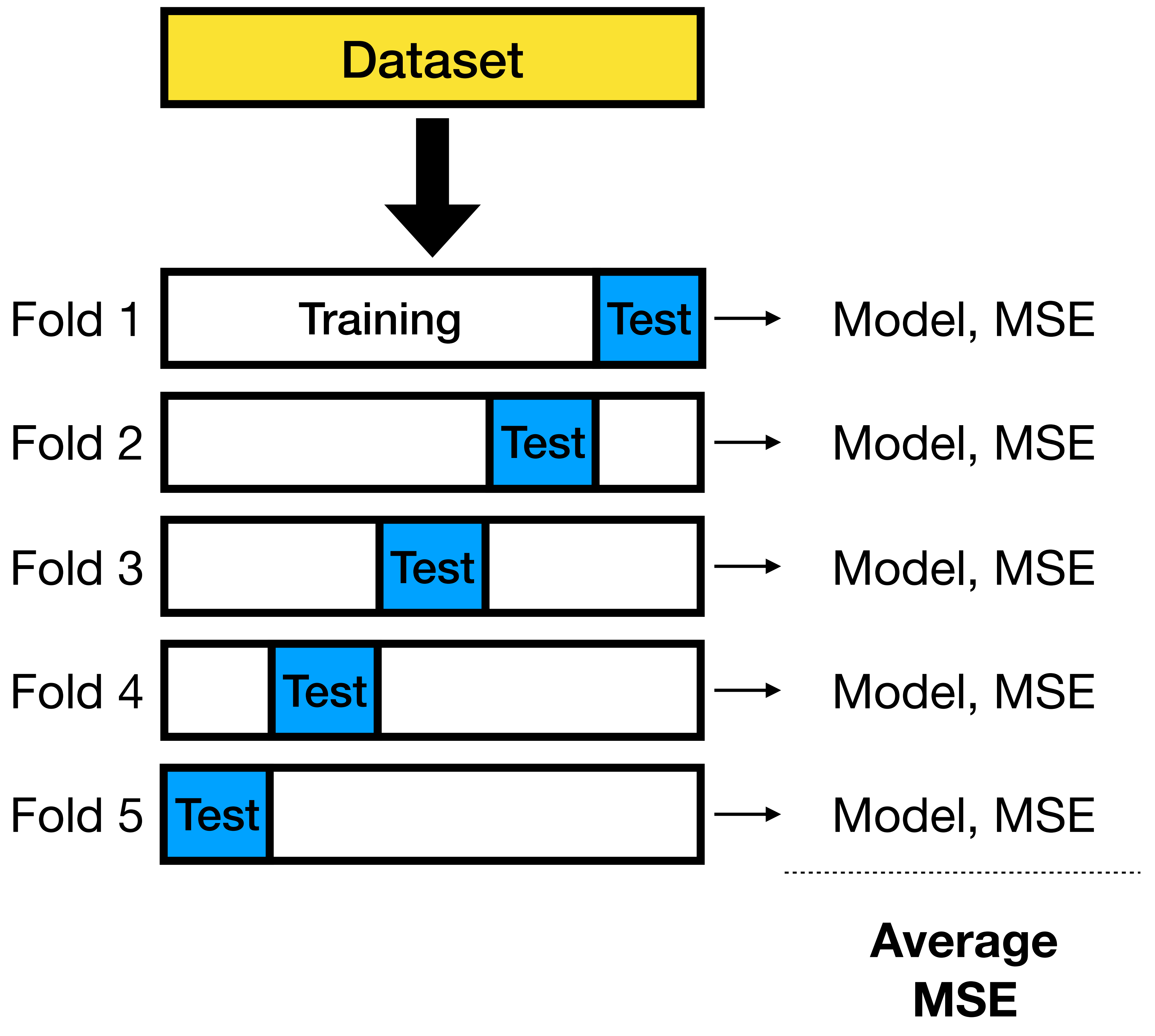
simulating testing data

- Ultimately, we'd like to actually test the model in the real world (e.g., predict tomorrow's temperature)
- However, this is usually quite costly, time consuming, or downright impossible, so we have to simulate it
- To do this, we can *split* our dataset into:
  - **Training data:** A subset we use to train/fit the model
  - **Testing data:** A subset we used to report the generalized performance
  - Common splits: 90/10 (i.e., 90% training and 10% test) and 80/20
- *Note*: It is important that the algorithm never sees the testing data (just like it is important that students don't see the real midterm)



cross validation

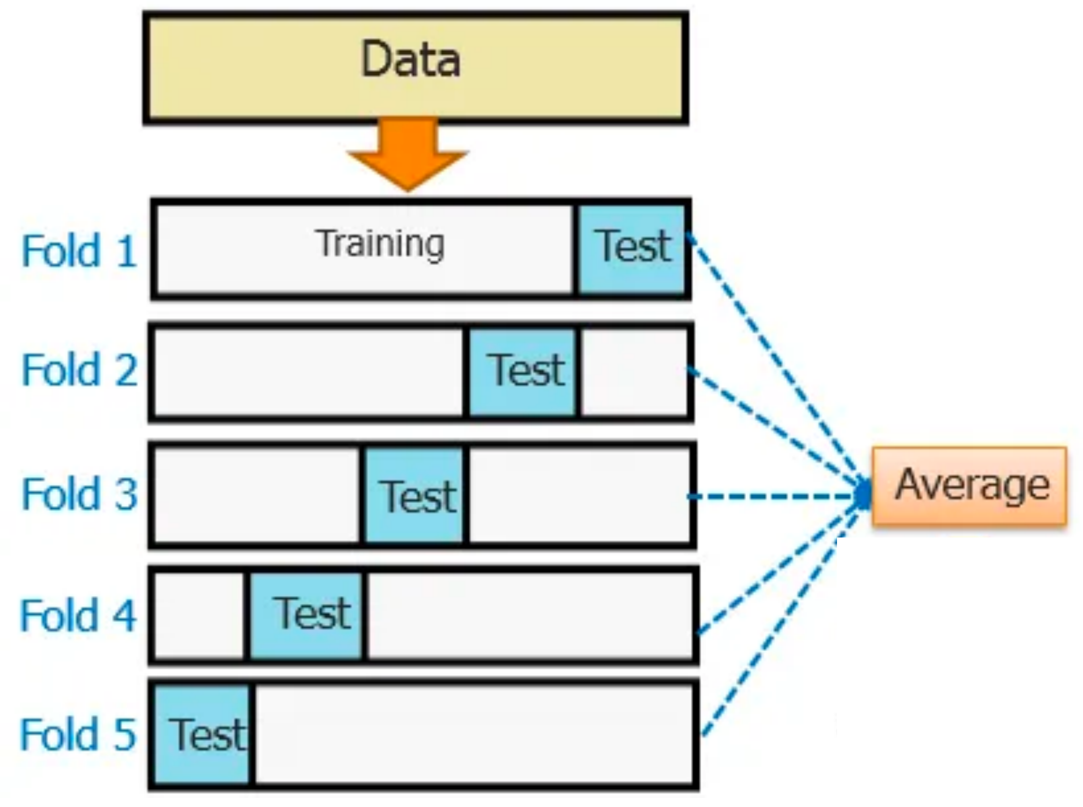
- k**-fold cross validation** (often abbreviated **CV**) repeats the train/test split idea k times, across different **folds** of the data
    - The data is divided into k parts
    - In each fold, one part is used as the testing set, and the other $k-1$ are used as the training set
    - Thus, there are k models fit throughout this process, and we can average testing performance (and sometimes the coefficients)
- How many folds should be used?
    - 3-fold, 5-fold and 10-fold are common
    - **Leave-one-out CV**: k is the number of datapoints, i.e., one is held out in each fold (computationally expensive)



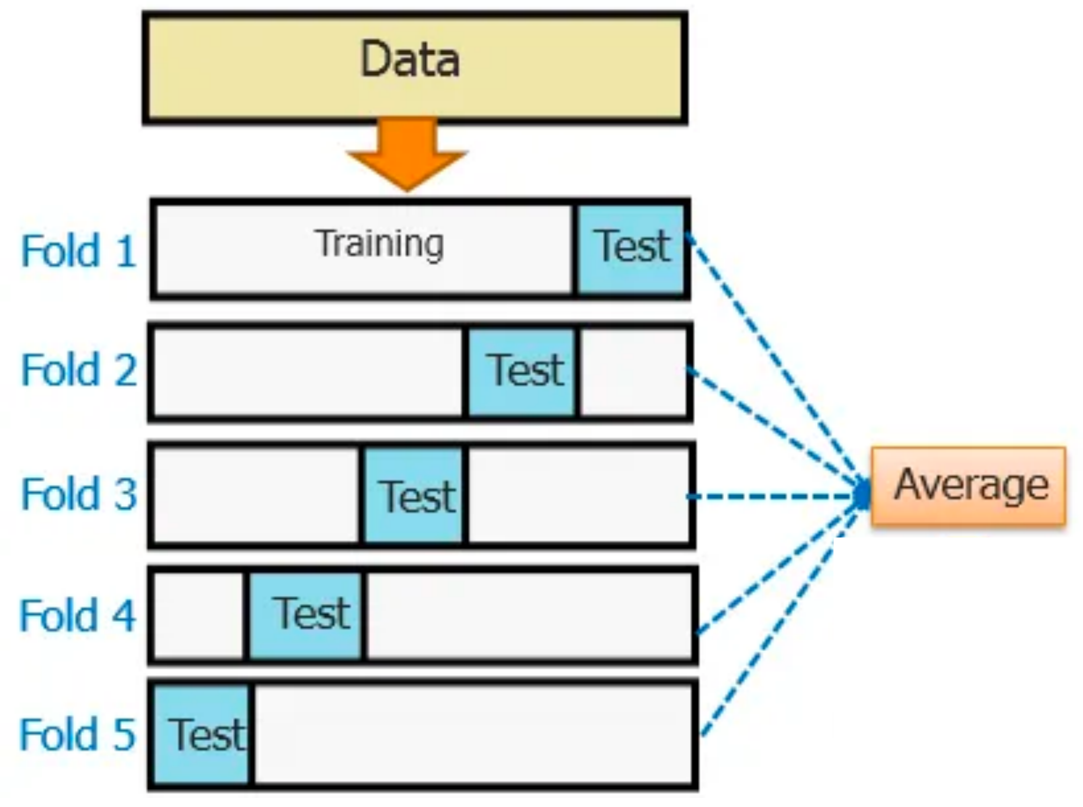
cross validation for model selection

- How do we determine the right value of λ?
- Test a wide range of λ typically log scale, e.g., 0.01,…,0.1,…,1,…,10,…,100
- Use multiple CV iterations, one for each value of λ:

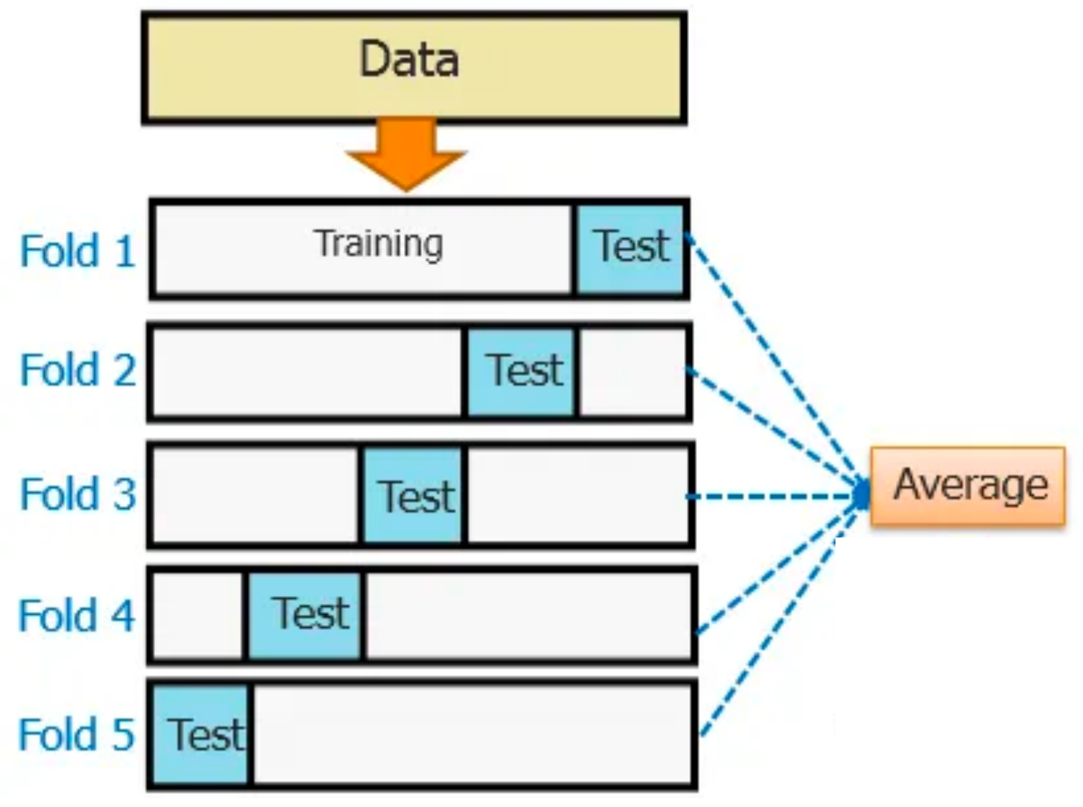
Train all folds with $\lambda = 0.01$



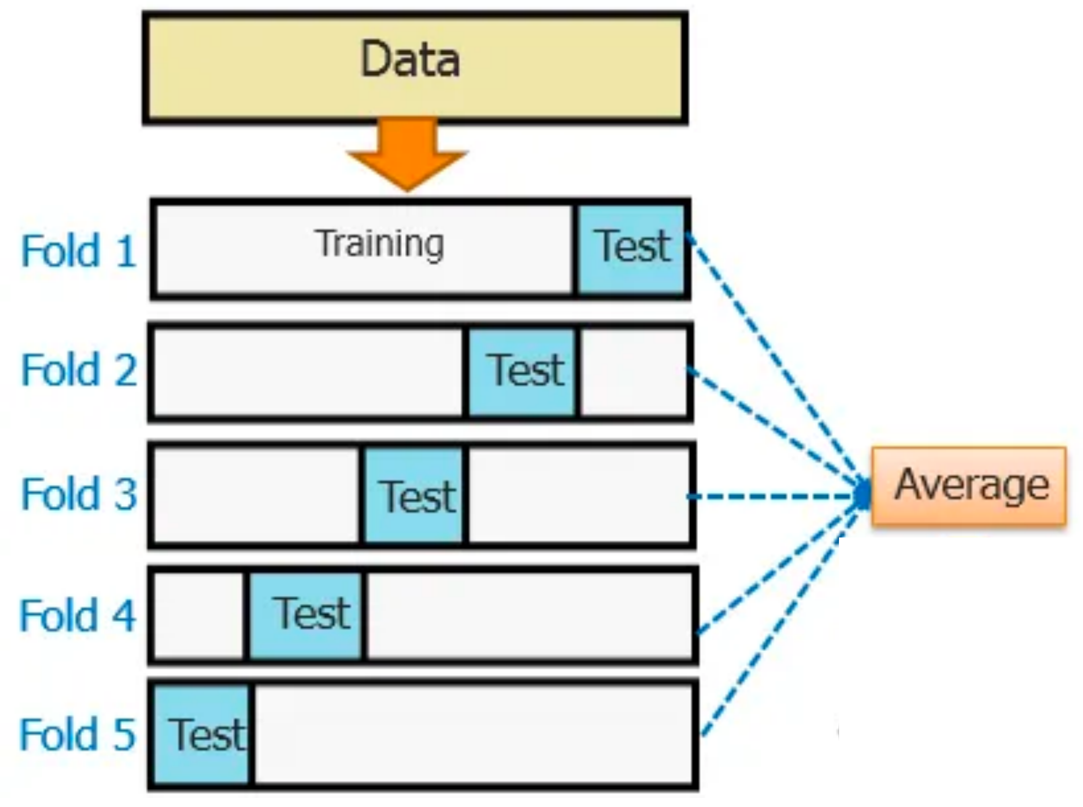
Train all folds with $\lambda = 0.1$



Train all folds with $\lambda = 1$



Train all folds with $\lambda = 10$



- Choose $\lambda^\star$ whose CV performance is the best
- For final model, train model with all data using $\lambda^\star$

$$\begin{tikzcd}\boxed{\text{Dataset}} \arrow[r] & \boxed{\text{Training with } \lambda^\star} \arrow[r] & \boxed{\text{Final model}}\end{tikzcd}$$

(very small) cv example

Suppose we collect three datapoints with a single feature x and target variable y. In the form (x, y), they are, approximately: (2.18, 2.26), (0.13, -14.57), (2.75, 16.74). Find the linear regression model $\hat{y} = ax + b$ and corresponding regularization parameter λ which has minimum cross validation error.

Use the Ridge model, $k = 3$ folds, and test $\lambda = 0,\ 0.1,\ 1$. Note that the coefficient b should NOT be regularized.

solution

- We need to solve the least squares equations for three values of lambda, and three folds each (i.e., 9 cases total). Here is the math for $\lambda = 0, 0.1$ and the second fold:

```
x ~ [2.18, 0.13, 2.75]
y ~ [2.26, -14.57, 16.74]
```

```
fold=2, lambda=0.0
X:
[[2.17997451 1.        ]
 [2.74831239 1.        ]]
X.T @ X:
[[12.30550986  4.9282869 ]
 [ 4.9282869   2.        ]]
X.T @ X + lambda*I:
[[12.30550986  4.9282869 ]
 [ 4.9282869   2.        ]]
(X.T @ X + lambda*I)^(-1):
[[  6.19179817 -15.25747891]
 [-15.25747891  38.09661673]]
(X.T @ X + lambda*I)^(-1)@ X^T:
[[-1.75951672  1.75951672]
 [ 4.8357016  -3.8357016 ]]
(X.T @ X + lambda*I)^(-1)@ X^T @ y:
[ 25.47215001 -53.26685674]
```

Only coefficient is changed by λ, intercept is not regularized

Notice how different the inverse is just from a small λ

```
fold=2, lambda=0.1
X:
[[2.17997451 1.        ]
 [2.74831239 1.        ]]
X.T @ X:
[[12.30550986  4.9282869 ]
 [ 4.9282869   2.        ]]
X.T @ X + lambda*I:
[[12.40550986  4.9282869 ]
 [ 4.9282869   2.        ]]
(X.T @ X + lambda*I)^(-1):
[[ 3.82403369 -9.42296757]
 [-9.42296757 23.71954383]]
(X.T @ X + lambda*I)^(-1)@ X^T:
[[-1.0866716   1.0866716]
 [ 3.1777147 -2.1777147]]
(X.T @ X + lambda*I)^(-1)@ X^T @ y:
[ 15.73151403 -29.26453239]
```

solution

```
x = [2.18, 0.13, 2.75]
y = [2.26, -14.57, 16.74]
```

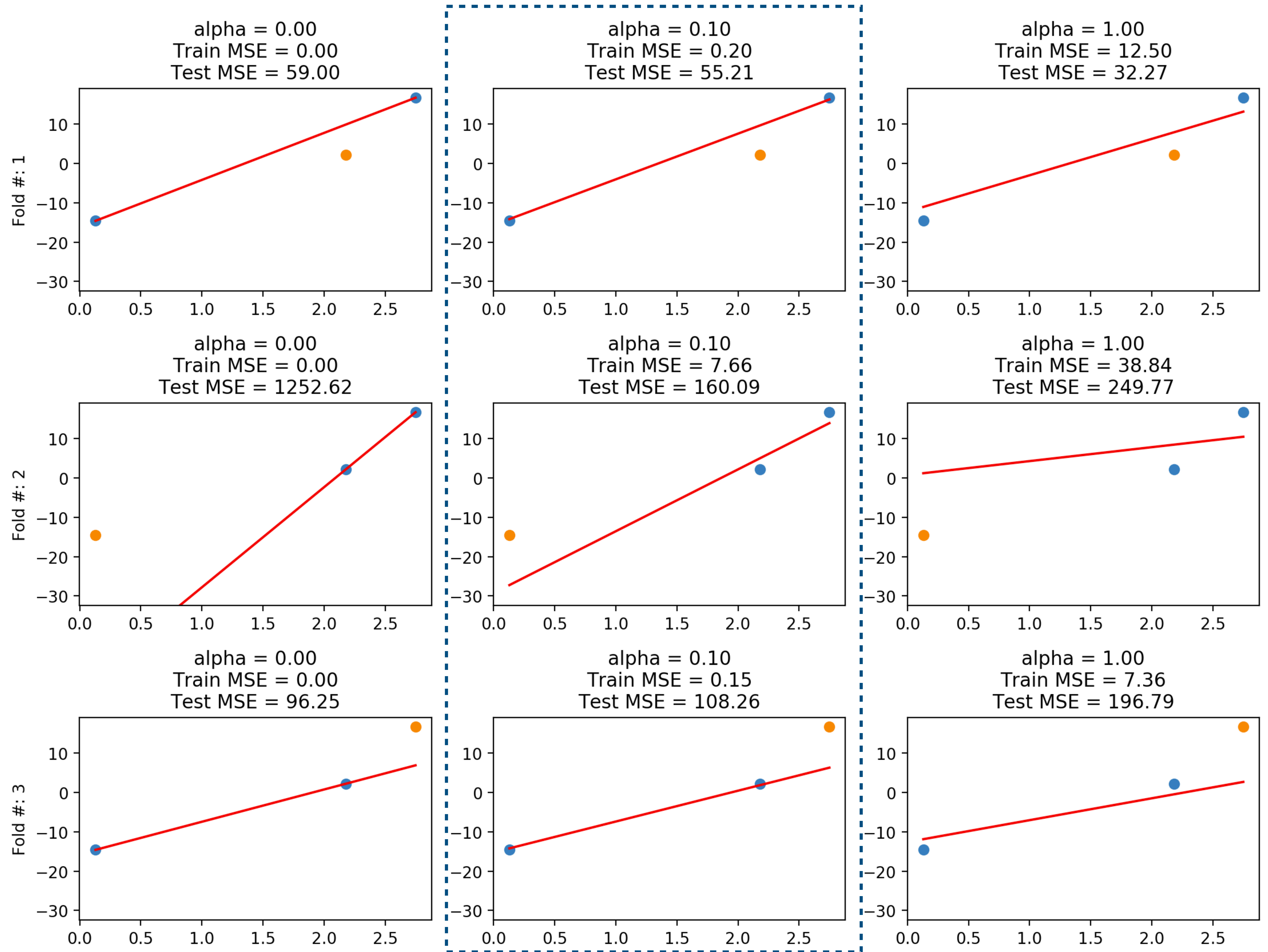
fold=2, lambda=0.0

```
X:
[[2.17997451 1.        ]
 [2.74831239 1.        ]]
X.T @ X:
[[12.30550986  4.9282869 ]
 [ 4.9282869   2.        ]]
X.T @ X + lambda*I:
[[12.30550986  4.9282869 ]
 [ 4.9282869   2.        ]]
(X.T @ X + lambda*I)^(-1):
[[  6.19179817 -15.25747891]
 [-15.25747891  38.09661673]]
(X.T @ X + lambda*I)^(-1)@ X^T:
[[-1.75951672  1.75951672]
 [ 4.8357016  -3.8357016 ]]
(X.T @ X + lambda*I)^(-1)@ X^T @ y:
[ 25.47215001 -53.26685674]
```

fold=2, lambda=0.1

```
X:
[[2.17997451 1.        ]
 [2.74831239 1.        ]]
X.T @ X:
[[12.30550986  4.9282869 ]
 [ 4.9282869   2.        ]]
X.T @ X + lambda*I:
[[12.40550986  4.9282869 ]
 [ 4.9282869   2.        ]]
(X.T @ X + lambda*I)^(-1):
[[ 3.82403369 -9.42296757]
 [-9.42296757 23.71954383]]
(X.T @ X + lambda*I)^(-1)@ X^T:
[[-1.0866716   1.0866716]
 [ 3.1777147 -2.1777147]]
(X.T @ X + lambda*I)^(-1)@ X^T @ y:
[ 15.73151403 -29.26453239]
```

$\lambda^\star = 0.10$ **has best average test MSE**


alpha = 0.00
Train MSE = 0.00
Test MSE = 59.00
alpha = 0.10
Train MSE = 0.20
Test MSE = 55.21
alpha = 1.00
Train MSE = 12.50
Test MSE = 32.27
alpha = 0.00
Train MSE = 0.00
Test MSE = 1252.62
alpha = 0.10
Train MSE = 7.66
Test MSE = 160.09
alpha = 1.00
Train MSE = 38.84
Test MSE = 249.77
alpha = 0.00
Train MSE = 0.00
Test MSE = 96.25
alpha = 0.10
Train MSE = 0.15
Test MSE = 108.26
alpha = 1.00
Train MSE = 7.36
Test MSE = 196.79
Fold #: 1
Fold #: 2
Fold #: 3